

HEARTS OF TWIN COTORSION PAIRS ON EXTRIANGULATED CATEGORIES

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ABSTRACT. In this article, we study the *heart* of a cotorsion pairs on an exact category and a triangulated category in a unified method, by means of the notion of an *extriangulated category*. We prove that the heart is abelian, and construct a cohomological functor to the heart. If the extriangulated category has enough projectives, this functor gives an equivalence between the heart and the category of coherent functors over the *coheart* modulo projectives. We also show how an *n-cluster tilting subcategory* of an extriangulated category gives rise to a family of cotorsion pairs with equivalent hearts.

CONTENTS

1. Introduction and Preliminaries	1
2. Hearts of twin cotorsion pairs	7
2.1. Adjoint property	9
2.2. The heart is semi-abelian	14
2.3. Functor to the heart	17
3. Hearts of cotorsion pairs	20
3.1. The heart is abelian	20
3.2. Associated cohomological functor	21
3.3. Heart-equivalence	23
4. Hearts with enough projectives	25
4.1. Condition for the heart to have enough projectives	25
4.2. Equivalence with the category of coherent functors	28
5. Relation with <i>n</i> -cluster tilting subcategories	29
5.1. Higher extensions	30
5.2. Cotorsion pairs induced from <i>n</i> -cluster tilting subcategories	31
References	35

1. INTRODUCTION AND PRELIMINARIES

The notion of a *cotorsion pair* goes back to [S]. It has been defined originally in module categories, and then in abelian and exact categories. Cotorsion pairs give a perspective of resolutions in the given category, and are related with other homological notions which are interesting in the category theory, or in the representation theory. For example, a pair of cotorsion pairs satisfying some conditions, which we call a *twin cotorsion pair*, has close relationship with model structure called *abelian model structure* ([Ho1],[Ho2]) and *exact model structure* [G].

If one considers a cotorsion pair in a triangulated category, it becomes essentially the same as the notion of a torsion theory in the sense of [IY], which has been introduced in the recent development of the theory of mutation, and of higher Auslander-Reiten theory. It gives a simultaneous generalization of *t*-structure [BBD], cluster tilting subcategory [KZ], co-*t*-structure, and is also related to notions such as *n*-cluster tilting subcategory. Quite often, an argument on cotorsion pairs which works in an exact

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category has its counterpart in a triangulated category through some appropriate modification, and vice versa. As for the relation with model structures mentioned above, its analog in a triangulated category has been given by [Y].

Recently, the notion of an *extriangulated category* was introduced in [NP], which is a simultaneous generalization of exact category and triangulated category. This enables us to unify the arguments on cotorsion pairs, as demonstrated in [NP] for the relation with model structures. Amongst several notions related to cotorsion pairs, this article is especially devoted to a unified treatment of the *hearts* of cotorsion pairs, by means of an extriangulated category.

Historically, the heart of a t-structure has been introduced in [BBD] to define perverse sheaves. More recently, the quotient category of a triangulated category by a cluster tilting subcategory has been used in the theory of cluster category [KZ, DL]. As a simultaneous generalization of them, the notions of hearts of cotorsion pairs were introduced in [N1] for a triangulated category, and in [L1] for an exact category. In both cases, the hearts are shown to become abelian categories, and they have similar properties.

In the latter part of this section, we briefly recall the definition and some basic properties of an extriangulated category, which will be used throughout this article. Although our main concern is about cotorsion pairs, we also deal with *twin* cotorsion pairs in section 2. The definition of the heart and its basic properties are given there. Especially, we will show that the heart of a twin cotorsion pair is always semi-abelian (Theorem 2.32). Since we regard a cotorsion pair as a special case of a twin cotorsion pair (Definition 2.3), which we call a *single* cotorsion pair when we emphasize, the results in section 2 will be used in the proceeding sections.

From section 3, we mainly deal with single cotorsion pairs. We specialize the results in section 2, to show that the heart of a cotorsion pair becomes abelian (Theorem 3.2), and that the associated functor to the heart becomes *cohomological* (Theorem 3.5). These unify the results in [N1], [L1] and [AN], [L2] respectively. We remark that we do not need the assumption of enough projectivity nor injectivity until this section, which gives a slight improvement of [L1] even in the case of an exact category. In the latter part, we also give a criterion for the hearts of cotorsion pairs to be equivalent compatibly with the associated cohomological functors (Proposition 3.12). It turns out that the *kernel* \mathcal{K} of a cotorsion pair $(\mathcal{U}, \mathcal{V})$ plays an important role.

From section 4, we assume that the extriangulated category has enough projectives. Under this assumption, we show that a pair associated to \mathcal{K} forms a cotorsion pair if and only if the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives of certain type (Theorem 4.10). We also give an equivalence between the heart and the category of coherent functors over the *coheart* modulo projectives (Proposition 4.15). The notion of a coheart is a generalization of that for a co-t-structure.

If we also assume the existence of enough injectives, we obtain a natural notion of higher extensions. This allows us to define *n-cluster tilting subcategories* of an extriangulated category. It gives a simultaneous generalization of the case of a triangulated category, and of an exact category. In section 5, we show how an *n*-cluster tilting subcategory induces a sequence of cotorsion pairs (Theorem 5.14). Combining the results in the preceding sections, we will see that all these have equivalent hearts, which are equivalent to some category of coherent functors (Corollary 5.15).

Let us briefly recall the definition and basic properties of extriangulated categories from [NP]. Throughout this article, let \mathcal{B} be an additive category.

Definition 1.1. Suppose \mathcal{B} is equipped with an additive bifunctor $\mathbb{E}: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \text{Ab}$. For any pair of objects $A, C \in \mathcal{B}$, an element $\delta \in \mathbb{E}(C, A)$ is called an \mathbb{E} -extension.

Remark 1.2. Let $\delta \in \mathbb{E}(C, A)$ be any \mathbb{E} -extension. By the functoriality, for any $a \in \mathcal{B}(A, A')$ and $c \in \mathcal{B}(C', C)$, we have \mathbb{E} -extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviately denote them by $a_*\delta$ and $c^*\delta$. In this terminology, we have

$$\mathbb{E}(c, a)(\delta) = c^*a_*\delta = a_*c^*\delta$$

in $\mathbb{E}(C', A')$.

Definition 1.3. Let $\delta \in \mathbb{E}(C, A)$, $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions. A morphism $(a, c): \delta \rightarrow \delta'$ of \mathbb{E} -extensions is a pair of morphisms $a \in \mathcal{B}(A, A')$ and $c \in \mathcal{B}(C, C')$ in \mathcal{B} , satisfying the equality

$$a_*\delta = c^*\delta'.$$

We simply denote it as $(a, c): \delta \rightarrow \delta'$.

Definition 1.4. For any $A, C \in \mathcal{B}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the split \mathbb{E} -extension.

Definition 1.5. Let $\delta = (A, \delta, C)$, $\delta' = (A', \delta', C')$ be any pair of \mathbb{E} -extensions. Let

$$C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C'$$

and

$$A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be coproduct and product in \mathcal{B} , respectively. Remark that, by the additivity of \mathbb{E} , we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through this isomorphism. This is the unique element which satisfies

$$\begin{aligned} \mathbb{E}(\iota_C, p_A)(\delta \oplus \delta') &= \delta \quad , \quad \mathbb{E}(\iota_C, p_{A'})(\delta \oplus \delta') = 0, \\ \mathbb{E}(\iota_{C'}, p_A)(\delta \oplus \delta') &= 0 \quad , \quad \mathbb{E}(\iota_{C'}, p_{A'})(\delta \oplus \delta') = \delta'. \end{aligned}$$

Definition 1.6. Let $A, C \in \mathcal{B}$ be any pair of objects. Sequences of morphisms in \mathcal{B}

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be equivalent if there exists an isomorphism $b \in \mathcal{B}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} & & B & & \\ & x \nearrow & \downarrow \cong & \searrow y & \\ A & & B & & C \\ & x' \searrow & \downarrow b & \nearrow y' & \\ & & B' & & \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Definition 1.7.

(1) For any $A, C \in \mathcal{B}$, we denote as

$$0 = [A \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} A \oplus C \xrightarrow{[0 \ 1]} C].$$

(2) For any $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we denote as

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 1.8. Let \mathfrak{s} be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. This \mathfrak{s} is called a realization of \mathbb{E} , if it satisfies the following condition (*). In this case, we say that sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ , whenever it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$.

(*) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$, $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then, for any morphism $(a, c): \delta \rightarrow \delta'$, there exists $b \in \mathcal{B}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \tag{1.1}$$

In the above situation, we say that the triplet (a, b, c) realizes (a, c) .

Definition 1.9. Let \mathcal{B}, \mathbb{E} be as above. A realization of \mathbb{E} is said to be additive, if it satisfies the following conditions.

(i) For any $A, C \in \mathcal{B}$, the split \mathbb{E} -extension $0 \in \mathbb{E}(C, A)$ satisfies

$$\mathfrak{s}(0) = 0.$$

(ii) For any pair of \mathbb{E} -extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$,

$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$$

holds.

Definition 1.10. ([NP, Definition 2.12]) A triplet $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is called an extriangulated category if it satisfies the following conditions.

(ET1) $\mathbb{E}: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \text{Ab}$ is an additive bifunctor.

(ET2) \mathfrak{s} is an additive realization of \mathbb{E} .

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, realized as

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \quad (1.2)$$

in \mathcal{B} , there exists a morphism $(a, c): \delta \rightarrow \delta'$ satisfying $cy = y'b$.

(ET3)^{op} Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of \mathbb{E} -extensions, realized by

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A' \xrightarrow{x'} B' \xrightarrow{y'} C'$$

respectively. For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in \mathcal{B} , there exists a morphism $(a, c): \delta \rightarrow \delta'$ satisfying $bx = x'a$.

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be \mathbb{E} -extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F$$

respectively. Then there exist an object $E \in \mathcal{B}$, a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D & & \\ & & \downarrow g & & \downarrow d & & \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E & & \\ & & \downarrow g' & & \downarrow e & & \\ & & F & \xlongequal{\quad} & F & & \end{array} \quad (1.3)$$

in \mathcal{B} , and an \mathbb{E} -extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,

- (ii) $d^*\delta'r = \delta$,
 (iii) $f_*\delta'r = e^*\delta'$.
 (ET4)^{op} Dual of (ET4).

Example 1.11. *Exact categories and triangulated categories are extriangulated categories. See [NP] for the detail.*

We use the following terminology.

Definition 1.12. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).*

- (1) *A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a conflation if it realizes some \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$. In this article, we write the conflation as $A \xrightarrow{x} B \xrightarrow{f} C$.*
- (2) *A morphism $f \in \mathcal{B}(A, B)$ is called an inflation if it admits some conflation $A \xrightarrow{x} B \xrightarrow{f} C$.*
- (3) *A morphism $f \in \mathcal{B}(A, B)$ is called a deflation if it admits some conflation $K \xrightarrow{y} A \xrightarrow{f} B$.*

Definition 1.13. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ as in Definition 1.12.*

- (1) *An object $P \in \mathcal{B}$ is called projective if it satisfies $\mathbb{E}(P, \mathcal{B}) = 0$. We denote the subcategory of projective objects by $\mathcal{P} \subseteq \mathcal{B}$. We say that \mathcal{B} has enough projectives if any object $B \in \mathcal{B}$ admits a deflation $P \rightarrow B$ from some $P \in \mathcal{P}$.*
- (2) *Dually, an object $I \in \mathcal{B}$ is called injective if it satisfies $\mathbb{E}(\mathcal{B}, I) = 0$. We denote the subcategory of injective objects by $\mathcal{I} \subseteq \mathcal{B}$. We say that \mathcal{B} has enough injectives if any object $B \in \mathcal{B}$ admits an inflation $B \rightarrow I$ to some $I \in \mathcal{I}$.*

Definition 1.14. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).*

- (1) *If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an \mathbb{E} -triangle, and write it in the following way.*

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A \quad (1.4)$$

- (2) *Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} A'$ be any pair of \mathbb{E} -triangles. If a triplet (a, b, c) realizes $(a, c): \delta \rightarrow \delta'$ as in (1.1), then we write it as*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} A \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} A' \end{array}$$

and call (a, b, c) a morphism of \mathbb{E} -triangles.

Definition 1.15. *Assume \mathcal{B} and \mathbb{E} satisfy (ET1). By Yoneda's lemma, any \mathbb{E} -extension $\delta \in \mathbb{E}(C, A)$ induces natural transformations*

$$\delta_{\#}: \mathcal{B}(-, C) \Rightarrow \mathbb{E}(-, A) \quad \text{and} \quad \delta^{\#}: \mathcal{B}(A, -) \Rightarrow \mathbb{E}(C, -).$$

For any $X \in \mathcal{B}$, these $(\delta_{\#})_X$ and $\delta_X^{\#}$ are given as follows.

- (1) $(\delta_{\#})_X: \mathcal{B}(X, C) \rightarrow \mathbb{E}(X, A); f \mapsto f^*\delta$.
- (2) $\delta_X^{\#}: \mathcal{B}(A, X) \rightarrow \mathbb{E}(C, X); g \mapsto g_*\delta$.

We abbreviately denote $(\delta_{\#})_X(f)$ and $\delta_X^{\#}(g)$ by $\delta_{\#}f$ and $\delta^{\#}g$, when there is no confusion.

Remark 1.16. By [NP, Corollary 3.8], for any \mathbb{E} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A$ and any $\delta' \in \mathbb{E}(C, A)$, the following are equivalent.

- (1) $\mathfrak{s}(\delta) = \mathfrak{s}(\delta')$.
- (2) There are automorphisms $a \in \mathcal{B}(A, A), c \in \mathcal{B}(C, C)$ satisfying $xa = x, cy = y$ and $\delta' = a_*c^*\delta$.

Definition 1.17. Let $\mathcal{D} \subseteq \mathcal{B}$ be a full additive subcategory, closed under isomorphisms. We say \mathcal{D} is extension-closed if it satisfies the following condition.

- If a conflation $A \rightarrow B \rightarrow C$ satisfies $A, C \in \mathcal{D}$, then $B \in \mathcal{D}$.

For two subcategories $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{B}$, we denote by $\mathcal{D}_1 * \mathcal{D}_2$ the subcategory which consists of the objects X admitting a conflation $D_1 \rightarrow X \rightarrow D_2$ for some $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$. In this notation, the above condition can be written as $\mathcal{D} * \mathcal{D} \subseteq \mathcal{D}$.

In the rest of this article, we fix an extriangulated category $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$. The following have been shown in [NP, Propositions 3.3, 3.11, 3.15].

Fact 1.18. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \dashrightarrow$ be any \mathbb{E} -triangle. Then the following sequences of natural transformations are exact.

$$\mathcal{B}(C, -) \xrightarrow{\circ y} \mathcal{B}(B, -) \xrightarrow{\circ x} \mathcal{B}(A, -) \xrightarrow{\delta^\sharp} \mathbb{E}(C, -) \xrightarrow{y^*} \mathbb{E}(B, -) \xrightarrow{x^*} \mathbb{E}(A, -),$$

$$\mathcal{B}(-, A) \xrightarrow{x \circ -} \mathcal{B}(-, B) \xrightarrow{y \circ -} \mathcal{B}(-, C) \xrightarrow{\delta_\sharp} \mathbb{E}(-, A) \xrightarrow{x_*} \mathbb{E}(-, B) \xrightarrow{y_*} \mathbb{E}(-, C).$$

Fact 1.19. The following holds.

- (1) Let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1} \dashrightarrow$ and $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2} \dashrightarrow$ be any pair of \mathbb{E} -triangles. Then there is a commutative diagram in \mathcal{B}

$$\begin{array}{ccccc} & & A_2 & \xlongequal{\quad} & A_2 \\ & & \downarrow m_2 & \circlearrowleft & \downarrow x_2 \\ A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\ \parallel & \circlearrowleft & \downarrow e_2 & \circlearrowleft & \downarrow y_2 \\ A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \end{array} \quad (1.5)$$

which satisfies

$$\begin{aligned} \mathfrak{s}(y_2^* \delta_1) &= [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2], \\ \mathfrak{s}(y_1^* \delta_2) &= [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1], \\ m_{1*} \delta_1 + m_{2*} \delta_2 &= 0. \end{aligned}$$

- (2) Dual of (1).

The following lemma is another version of [NP, Corollary 3.16].

Proposition 1.20. Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \dashrightarrow$ be any \mathbb{E} -triangle, let $f: A \rightarrow D$ be any morphism, and let $D \xrightarrow{d} E \xrightarrow{e} C \xrightarrow{f_* \delta} \dashrightarrow$ be any \mathbb{E} -triangle realizing $f_* \delta$. Then there is a morphism g which gives a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \dashrightarrow \\ f \downarrow & & \downarrow g & & \parallel \\ D & \xrightarrow{d} & E & \xrightarrow{e} & C \xrightarrow{f_* \delta} \dashrightarrow \end{array} \quad (1.6)$$

and moreover, $A \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g \end{pmatrix}} E \xrightarrow{e^* \delta} \dashrightarrow$ becomes an \mathbb{E} -triangle.

Proof. By Fact 1.19, we get the following commutative diagram made of \mathbb{E} -triangles

$$\begin{array}{ccccc}
 & A & \xlongequal{\quad} & A & \\
 & \downarrow \begin{pmatrix} \exists h \\ x \end{pmatrix} & & \downarrow x & \\
 D & \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} & D \oplus B & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & B \xrightarrow{0} \gg \\
 \parallel & & \downarrow \begin{pmatrix} d & \exists g' \end{pmatrix} & & \downarrow y \\
 D & \xrightarrow{d} & E & \xrightarrow{e} & C \xrightarrow{f_*\delta} \gg \\
 & & \downarrow e^*\delta & & \downarrow \delta \\
 & & \Psi & & \Psi
 \end{array}$$

satisfying $\begin{pmatrix} h \\ x \end{pmatrix}_* \delta + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_* f_* \delta = 0$, in which we may assume that the middle row is of the form $D \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} B \xrightarrow{0} \gg$ since we have $y^* f_* \delta = f_* y^* \delta = 0$. In particular, $A \xrightarrow{\begin{pmatrix} h \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g' \end{pmatrix}} E \xrightarrow{e^* \delta} \gg$ is an \mathbb{E} -triangle. By the above equality, we have

$$(f + h)_* \delta = \begin{pmatrix} 1 & 0 \end{pmatrix}_* \left(\begin{pmatrix} h \\ x \end{pmatrix}_* \delta + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_* f_* \delta \right) = 0$$

in $\mathbb{E}(C, D)$. Thus by the exactness of $\mathcal{B}(B, D) \xrightarrow{-\circ x} \mathcal{B}(A, D) \xrightarrow{\delta^\#} \mathbb{E}(C, D)$, there is $b \in \mathcal{B}(B, D)$ which gives $f + h = bx$.

For the isomorphism $i = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} : D \oplus B \xrightarrow{\cong} D \oplus B$, the following diagram is commutative.

$$\begin{array}{ccccc}
 & & D \oplus B & & \\
 & \nearrow \begin{pmatrix} h \\ x \end{pmatrix} & \downarrow \cong & \nwarrow \begin{pmatrix} d & g' \end{pmatrix} & \\
 A & & & & E \\
 & \searrow \begin{pmatrix} -f \\ x \end{pmatrix} & \downarrow i & \nearrow \begin{pmatrix} d & db + g' \end{pmatrix} & \\
 & & D \oplus B & &
 \end{array}$$

Thus if we put $g = db + g'$, then it gives an \mathbb{E} -triangle $A \xrightarrow{\begin{pmatrix} -f \\ x \end{pmatrix}} D \oplus B \xrightarrow{\begin{pmatrix} d & g \end{pmatrix}} E \xrightarrow{e^* \delta} \gg$. Commutativity of (1.6) follows from $gx = (db + g')x = df + (dh + g'x) = df$ and $eg = e(db + g') = edb + eg' = eg' = g$. \square

2. HEARTS OF TWIN COTORSION PAIRS

As before, $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ denotes an extriangulated category.

Definition 2.1. Let \mathcal{U} and \mathcal{V} be full additive subcategories of \mathcal{B} which are closed under direct summands and isomorphisms. We call $(\mathcal{U}, \mathcal{V})$ a cotorsion pair if it satisfies the following conditions. As below, we always require a cotorsion pair to be complete, in the sense of [Ho2].

- (a) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$.
- (b) $(\mathcal{U}, \mathcal{V})$ is complete. Namely, for any object $B \in \mathcal{B}$, there exist two conflations

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$.

A cotorsion pair $(\mathcal{U}, \mathcal{V})$ is said to be rigid if it satisfies $\mathcal{U} \subseteq \mathcal{V}$.

Remark 2.2. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on \mathcal{B} , the following holds.

- (a) A morphism $f : A \rightarrow B$ factors through \mathcal{U} if and only if $\mathbb{E}(f, \mathcal{V}) = 0$.
- (b) A morphism $f : A \rightarrow B$ factors through \mathcal{V} if and only if $\mathbb{E}(\mathcal{U}, f) = 0$.
- (c) \mathcal{U} and \mathcal{V} are closed under extension.
- (d) $\mathcal{P} \subseteq \mathcal{U}$ and $\mathcal{I} \subseteq \mathcal{V}$.

Definition 2.3. A pair of cotorsion pairs $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on \mathcal{B} is called a twin cotorsion pair if it satisfies $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$, or equivalently $\mathcal{S} \subseteq \mathcal{U}$.

To distinguish from a twin cotorsion pair, we sometimes call cotorsion pair $(\mathcal{U}, \mathcal{V})$ a single cotorsion pair. Remark that any cotorsion pair $(\mathcal{U}, \mathcal{V})$ gives a twin cotorsion pair $((\mathcal{U}, \mathcal{V}), (\mathcal{U}, \mathcal{V}))$. Thus a cotorsion pair can be regarded as a special case of a twin cotorsion pair, satisfying $\mathcal{S} = \mathcal{U}$ and $\mathcal{T} = \mathcal{V}$. In this way, any argument on twin cotorsion pairs can be applied to cotorsion pairs.

Remark 2.4. If \mathcal{B} is triangulated or exact, then this definition agrees with those in [N2] and [L1], respectively.

Definition 2.5. For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, put $\mathcal{W} = \mathcal{T} \cap \mathcal{U}$ and call it the core of $(\mathcal{U}, \mathcal{V})$. We define as follows.

- (a) $\mathcal{B}^+ = \text{Cone}(\mathcal{V}, \mathcal{W})$. Namely, \mathcal{B}^+ is defined to be the full subcategory of \mathcal{B} , consisting of objects B which admits a conflation

$$V_B \rightarrowtail W_B \twoheadrightarrow B$$

where $W_B \in \mathcal{W}$ and $V_B \in \mathcal{V}$. It can be easily shown that we have $\mathcal{T} \subseteq \mathcal{B}^+$.

- (b) $\mathcal{B}^- = \text{CoCone}(\mathcal{W}, \mathcal{S})$. Namely, \mathcal{B}^- is defined to be the full subcategory of \mathcal{B} , consisting of objects B which admits a conflation

$$B \rightarrowtail W^B \twoheadrightarrow S^B$$

where $W^B \in \mathcal{W}$ and $S^B \in \mathcal{S}$. It can be easily shown that we have $\mathcal{U} \subseteq \mathcal{B}^-$.

Definition 2.6. Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on \mathcal{B} , and write the quotient of \mathcal{B} by \mathcal{W} as $\underline{\mathcal{B}} = \mathcal{B}/\mathcal{W}$. For any morphism $f \in \mathcal{B}(X, Y)$, we denote its image in $\underline{\mathcal{B}}(X, Y)$ by \underline{f} .

For any full additive subcategory \mathcal{C} of \mathcal{B} containing \mathcal{W} , similarly we put $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{W}$. This is a full subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as \mathcal{C} .

Put $\mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^-$. Since $\mathcal{H} \supseteq \mathcal{W}$, we obtain a full additive subcategory $\underline{\mathcal{H}} \subseteq \underline{\mathcal{B}}$, which we call the heart of the twin cotorsion pair.

Remark 2.7. By using Fact 1.18, we can easily confirm $\underline{\mathcal{B}}(\underline{\mathcal{U}}, \underline{\mathcal{B}}^+) = 0$ and $\underline{\mathcal{B}}(\underline{\mathcal{B}}^-, \underline{\mathcal{T}}) = 0$.

Here are some properties of \mathcal{B}^+ and \mathcal{B}^- . The following is a corollary of Proposition 1.20.

Lemma 2.8. Let $f \in \mathcal{B}(A, B)$ be any morphism.

- (1) If $B \in \mathcal{B}^+$, then there exist $W \in \mathcal{W}$ and $w \in \mathcal{B}(W, B)$ which give a deflation $(f \ w) : A \oplus W \rightarrow B$.
- (2) Dually if $A \in \mathcal{B}^-$, then there exist $W' \in \mathcal{W}$ and $w' \in \mathcal{B}(A, W')$, which gives an inflation $\begin{pmatrix} f \\ w' \end{pmatrix} : A \rightarrow B \oplus W'$.

Proof. (1) Since $B \in \mathcal{B}^+$, it admits a conflation $V_B \rightarrowtail W_B \xrightarrow{w_B} B$ with $V_B \in \mathcal{V}, W_B \in \mathcal{W}$. By the dual of Proposition 1.20, we get a conflation $C \twoheadrightarrow A \oplus W_B \xrightarrow{(f \ -w_B)} B$. (2) can be shown dually. \square

Lemma 2.9. Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be as before.

- (a) If $A \xrightarrow{f} B \xrightarrow{g} U$ is a conflation in \mathcal{B} with $U \in \mathcal{U}$, then $A \in \mathcal{B}^-$ implies $B \in \mathcal{B}^-$.
- (b) If $A \xrightarrow{f} B \xrightarrow{g} S$ is a conflation in \mathcal{B} with $S \in \mathcal{S}$, then $B \in \mathcal{B}^-$ implies $A \in \mathcal{B}^-$. In particular, this shows $\mathcal{B}^- = \text{CoCone}(\mathcal{U}, \mathcal{S})$.

Proof. (a) Since $A \in \mathcal{B}^-$, it admits an \mathbb{E} -triangle $A \xrightarrow{w^A} W^A \rightarrow S^A \dashrightarrow$, with $W^A \in \mathcal{W}$ and $S^A \in \mathcal{S}$. By $\mathbb{E}(\mathcal{S}, \mathcal{T}) = 0$, we have a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} A & \xrightarrow{w^A} & W^A & \longrightarrow & S^A \dashrightarrow \\ \downarrow f & & \downarrow & & \downarrow \\ B & \xrightarrow{t^B} & T^B & \longrightarrow & S^B \dashrightarrow \end{array}$$

with $S^B \in \mathcal{S}$ and $T^B \in \mathcal{T}$. Thus we obtain a commutative diagram

$$\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
 & \mathbb{E}(T^B, V) & \longrightarrow 0 \\
 (t^B)^* \downarrow & & \downarrow \\
 0 \longrightarrow \mathbb{E}(B, V) & \xrightarrow{f^*} & \mathbb{E}(A, V)
 \end{array}$$

for any $V \in \mathcal{V}$, in which the bottom row and the left column are exact. This shows $\mathbb{E}(T^B, \mathcal{V}) = 0$, and thus $T^B \in \mathcal{T} \cap \mathcal{U} = \mathcal{W}$.

(b) Since $B \in \mathcal{B}^-$, there exists a conflation $B \rightrightarrows^{w^B} W^B \rightrightarrows S^B$ with $S^B \in \mathcal{S}$ and $W^B \in \mathcal{W}$. By (ET4), we get a commutative diagram made of conflations as follows.

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & S \\
 \parallel & & \downarrow w^B & & \downarrow \\
 A & \xrightarrow{\quad} & W^B & \xrightarrow{\quad} & X \\
 & & \downarrow & & \downarrow \\
 & & S^B & \xlongequal{\quad} & S^B
 \end{array}$$

We thus get $X \in \mathcal{S}$ since $\mathcal{S} \subseteq \mathcal{B}$ is closed under extension. This shows $A \in \mathcal{B}^-$. \square

Dually, the following holds.

Lemma 2.10. *Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be as before.*

- (a) *If $T \rightrightarrows A \xrightarrow{f} B$ is a conflation in \mathcal{B} with $T \in \mathcal{T}$, then $B \in \mathcal{B}^+$ implies $A \in \mathcal{B}^+$.*
- (b) *If $V \rightrightarrows A \xrightarrow{f} B$ is a conflation in \mathcal{B} with $V \in \mathcal{V}$, then $A \in \mathcal{B}^+$ implies $B \in \mathcal{B}^+$. In particular, this shows $\mathcal{B}^+ = \text{Cone}(\mathcal{V}, \mathcal{T})$.*

2.1. Adjoint property. We fix a twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$.

Lemma 2.11. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{\mu} \gg$ and $C \xrightarrow{c} D \xrightarrow{d} E \xrightarrow{\nu} \gg$ be \mathbb{E} -triangles. The following are equivalent.*

- (1) *There exists $\theta \in \mathbb{E}(E, B)$ satisfying $b_*\theta = \nu$.*
- (2) *There exists $\tau \in \mathbb{E}(D, A)$ satisfying $c^*\tau = \mu$.*

Proof. Assume (1), and realize θ by $B \xrightarrow{x} X \xrightarrow{y} E \xrightarrow{\theta} \gg$. Then by (ET4), we obtain the following commutative diagram made of \mathbb{E} -triangles,

$$\begin{array}{ccccc}
 A & \xrightarrow{a} & B & \xrightarrow{b} & C \xrightarrow{\mu} \gg \\
 \parallel & & \downarrow x & & \downarrow \exists c' \\
 A & \xrightarrow{x \circ a} & X & \xrightarrow{\quad} & \exists D' \xrightarrow{\exists \gamma} \gg \\
 & & \downarrow y & & \downarrow \exists d' \\
 & & E & \xlongequal{\quad} & E \\
 & & \downarrow \theta & & \downarrow b_*\theta \\
 & & \mathbb{V} & & \mathbb{V}
 \end{array}$$

satisfying $c'^*\gamma = \mu$. Since $b_*\theta = \nu$, there is an isomorphism $i \in \mathcal{B}(D, D')$ satisfying $c' = i \circ c$ and $d' \circ i = d$. Then $i^*\gamma \in \mathbb{E}(D, A)$ satisfies $c^*(i^*\gamma) = c'^*\gamma = \mu$. \square

Proposition 2.12. *Let $B \xrightarrow{c} D \xrightarrow{d} E \dashrightarrow^\nu$ be \mathbb{E} -triangle, and let $V_B \xrightarrow{v_B} U_B \xrightarrow{u_B} B \dashrightarrow^\lambda$ be an \mathbb{E} -triangle satisfying $U_B \in \mathcal{U}, V_B \in \mathcal{V}$. Then the following are equivalent.*

- (1) $\mathbb{E}(D, V) \xrightarrow{c^*} \mathbb{E}(B, V)$ is surjective for any $V \in \mathcal{V}$.
- (2) $\mathbb{E}(D, V_B) \xrightarrow{c^*} \mathbb{E}(B, V_B)$ is surjective.
- (3) There exists $\tau \in \mathbb{E}(D, V_B)$ satisfying $c^*\tau = \lambda_B$.
- (4) There exists $\mu \in \mathbb{E}(E, U_B)$ satisfying $(u_B)_*\mu = \nu$.

Proof. (3) \Leftrightarrow (4) follows from Lemma 2.11. (1) \Rightarrow (2) \Rightarrow (3) is obvious. Let us show (3) \Rightarrow (1).

Take any $V \in \mathcal{V}$ and $\lambda \in \mathbb{E}(B, V)$. By the exactness of $\mathcal{B}(V_B, V) \xrightarrow{\lambda_B^\#} \mathbb{E}(B, V) \rightarrow 0$, there exists $f \in \mathcal{B}(V_B, V)$ satisfying $\lambda = f_*\lambda_B$. Then $f_*\tau \in \mathbb{E}(D, V)$ satisfies $c^*(f_*\tau) = f_*\lambda_B = \lambda$. \square

Corollary 2.13. *Let*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow^\delta \\ \downarrow a & & \downarrow b & & \parallel \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \dashrightarrow_{a_*\delta} \end{array}$$

be a morphism of \mathbb{E} -triangles. If $\mathbb{E}(B, V) \xrightarrow{x^} \mathbb{E}(A, V)$ is surjective for any $V \in \mathcal{V}$, then so is $\mathbb{E}(B', V) \xrightarrow{x'^*} \mathbb{E}(A', V)$ for any $V \in \mathcal{V}$.*

Proof. Resolve A and A' by \mathbb{E} -triangles

$$V_A \rightarrow U_A \xrightarrow{u_A} A \dashrightarrow, \quad V_{A'} \rightarrow U_{A'} \xrightarrow{u_{A'}} A' \dashrightarrow$$

satisfying $U_A, U_{A'} \in \mathcal{U}$ and $V_A, V_{A'} \in \mathcal{V}$. Then a induces a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc} V_A & \longrightarrow & U_A & \xrightarrow{u_A} & A \dashrightarrow \\ \exists \downarrow & & \downarrow \exists u & & \downarrow a \\ V_{A'} & \longrightarrow & U_{A'} & \xrightarrow{u_{A'}} & A' \dashrightarrow \end{array}$$

By Proposition 2.12, there exists $\mu \in \mathbb{E}(C, U_A)$ which gives $(u_A)_*\mu = \delta$. Then $u_*\mu \in \mathbb{E}(C, U_{A'})$ satisfies

$$(u_{A'})_*(u_*\mu) = a_*(u_A)_*\mu = a_*\delta.$$

Again by Proposition 2.12, $\mathbb{E}(B', V) \xrightarrow{x'^*} \mathbb{E}(A', V)$ becomes surjective for any $V \in \mathcal{V}$. \square

Definition 2.14. (a) *A conflation with $Z \in \mathcal{B}^+, S \in \mathcal{S}$*

$$B \rightrightarrows^z Z \twoheadrightarrow S$$

is called a reflection sequence for B , if $\mathbb{E}(Z, V) \xrightarrow{z^} \mathbb{E}(B, V)$ is surjective (and thus bijective) for any $V \in \mathcal{V}$.*

(b) *Dually, a conflation with $X \in \mathcal{B}^-, V \in \mathcal{V}$*

$$V \twoheadrightarrow X \rightrightarrows^x B$$

is called a coreflection sequence for B , if $\mathbb{E}(S, X) \xrightarrow{x^} \mathbb{E}(S, B)$ is surjective for any $S \in \mathcal{S}$.*

Proposition 2.15. *If $B \rightrightarrows^z Z \twoheadrightarrow S$ is a reflection sequence, then the following hold for any $Y \in \mathcal{B}^+$.*

- (1) $- \circ z: \mathcal{B}^+(Z, Y) \rightarrow \mathcal{B}(B, Y)$ is surjective.
- (2) $- \circ \underline{z}: \underline{\mathcal{B}}^+(Z, Y) \rightarrow \underline{\mathcal{B}}(B, Y)$ is bijective.

Dually for coreflection sequences.

Proof. By $Y \in \mathcal{B}^+$, there is an \mathbb{E} -triangle $V_Y \xrightarrow{v_Y} W_Y \xrightarrow{w_Y} Y \xrightarrow{\lambda_Y} \succ$ satisfying $V_Y \in \mathcal{V}_Y$ and $W_Y \in \mathcal{W}_Y$.

(1) Let $f \in \mathcal{B}(B, Y)$ be any morphism. Since $z^*: \mathbb{E}(Z, V_Y) \rightarrow \mathbb{E}(B, V_Y)$ is surjective, there is $\beta \in \mathbb{E}(Z, V_Y)$ satisfying $z^*\beta = f^*\lambda_Y$. Then the exactness of

$$\begin{aligned} \mathcal{B}(Z, Y) &\xrightarrow{(\lambda_Y)^\#} \mathbb{E}(Z, V_Y) \xrightarrow{(v_Y)^*} \mathbb{E}(Z, W_Y), \\ 0 \rightarrow \mathbb{E}(Z, W_Y) &\xrightarrow{z^*} \mathbb{E}(B, W_Y) \end{aligned}$$

and $z^*(v_Y)_*\beta = f^*(v_Y)_*\lambda_Y = 0$ shows the existence of $g \in \mathcal{B}(Z, Y)$ which gives $g^*\lambda_Y = \beta$.

By $(\lambda_Y)_\#(f - gz) = f^*\lambda_Y - z^*\beta = 0$, we obtain $h \in \mathcal{B}(B, W_Y)$ satisfying $f - gz = w_Yh$. By the exactness of

$$\mathcal{B}(Z, W_Y) \xrightarrow{\circ z} \mathcal{B}(B, W_Y) \rightarrow 0, \quad (2.1)$$

we have $i \in \mathcal{B}(Z, W_Y)$ which gives $iz = h$. Then $g + w_Yi \in \mathcal{B}(Z, W_Y)$ satisfies $f = (g + w_Yi)z$.

(2) Suppose $f \in \mathcal{B}^+(Z, Y)$ satisfies $\underline{f}z = 0$. This implies that there exists $g \in \mathcal{B}(B, W_Y)$ satisfying $w_Yg = fz$. By the exactness of (2.1), there is $h \in \mathcal{B}(Z, W_Y)$ which gives $hz = g$. Then by the exactness of

$$\mathcal{B}(S, Y) \rightarrow \mathcal{B}(Z, Y) \xrightarrow{\circ z} \mathcal{B}(B, Y)$$

and the equality $(f - w_Yh)z = 0$, it follows that $f - w_Yh$ factors through S . Since $\underline{\mathcal{B}}(S, Y) = 0$ by Remark 2.7, this means $\underline{f} = \underline{w_Yh} = 0$. \square

Corollary 2.16. *By Proposition 2.15, a reflection sequence $B \xrightarrow{z} Z \rightarrow S$ gives a reflection (Z, \underline{z}) of B along the inclusion functor $\underline{\mathcal{B}}^+ \hookrightarrow \underline{\mathcal{B}}$, in the sense of [Bo, Definition 3.1]. In particular, $Z \in \mathcal{B}^+$ is uniquely determined by B , up to isomorphism in $\underline{\mathcal{B}}^+$. Dually for coreflection sequences.*

Proof. This immediately follows from Proposition 2.15. \square

Lemma 2.17. *Let $A \in \mathcal{B}$ be any object, and let $B \xrightarrow{z} Z \rightarrow S$ be a reflection sequence. For any morphism $f \in \mathcal{B}(A, B)$, the following are equivalent.*

- (1) zf satisfies $\underline{zf} = 0$.
- (2) f factors some object in \mathcal{U} .

Proof. (2) \Rightarrow (1) follows from $\underline{\mathcal{B}}(\underline{\mathcal{U}}, \underline{\mathcal{B}}^+) = 0$, as in Remark 2.7. Let us show the converse.

Let $V_B \xrightarrow{v_B} U_B \xrightarrow{u_B} B \xrightarrow{\lambda_B} \succ$ be an \mathbb{E} -triangle satisfying $U_B \in \mathcal{U}$, $V_B \in \mathcal{V}$. By Proposition 2.12, there exists $\tau \in \mathbb{E}(Z, V_B)$ satisfying $z^*\tau = \lambda_B$. Then $\underline{zf} = 0$ and $\mathbb{E}(\mathcal{W}, V_B) = 0$ shows $f^*\lambda_B = (zf)^*\tau = 0$.

Thus (2) follows from the exactness of $\mathcal{B}(A, U_B) \xrightarrow{u_B \circ -} \mathcal{B}(A, B) \xrightarrow{(\lambda_B)^\#} \mathbb{E}(A, V_B)$. \square

The following gives a (co)reflection sequence for each $B \in \mathcal{B}$ (Proposition 2.19).

Definition 2.18. *Let $B \in \mathcal{B}$ be any object. We define as follows.*

- (1) Take two \mathbb{E} -triangles

$$V_B \rightarrow U_B \xrightarrow{u_B} B \xrightarrow{\delta} \succ, \quad U_B \rightarrow T^U \rightarrow S^U \dashrightarrow$$

where $U_B \in \mathcal{U}$, $V_B \in \mathcal{V}$, $T^U \in \mathcal{T}$ and $S^U \in \mathcal{S}$ which implies $T^U \in \mathcal{W}$. By [NP, (ET4)], we get the following commutative diagram.

$$\begin{array}{ccccc} V_B & \xrightarrow{\quad} & U_B & \xrightarrow{u_B} & B \xrightarrow{\delta} \succ \\ \parallel & & \downarrow u & & \downarrow p_B \\ V_B & \xrightarrow{\quad} & T^U & \xrightarrow{t} & B^+ \xrightarrow{\delta''} \succ \\ & & \downarrow & & \downarrow \\ & & S^U & = & S^U \end{array} \quad (2.2)$$

By Definition 2.5, we have $B^+ \in \mathcal{B}^+$.

(2) Dually, take the following two conflations

$$B \rightharpoonup T^B \twoheadrightarrow S^B, \quad V_T \rightharpoonup U_T \twoheadrightarrow T^B$$

where $U_T \in \mathcal{U}$, $V_T \in \mathcal{V}$, $T^B \in \mathcal{T}$ and $S^B \in \mathcal{S}$. By (ET4)^{op}, we get the following commutative diagram

$$\begin{array}{ccccc} V_T & \xlongequal{\quad} & V_T & & \\ \downarrow & & \downarrow & & \\ B^- & \xrightarrow{\quad} & U_T & \twoheadrightarrow & S^B \\ \downarrow m_B & & \downarrow & & \parallel \\ B & \xrightarrow{\quad} & T^B & \twoheadrightarrow & S^B \end{array} \quad (2.3)$$

in which, B^- belongs to \mathcal{B}^- .

Proposition 2.19. For any $B \in \mathcal{B}$, the following holds.

- (a) The conflation $B \xrightarrow{p_B} B^+ \twoheadrightarrow S^U$ in Definition 2.18 (1) is a reflection sequence for B .
- (b) The conflation $V_T \rightharpoonup B^- \xrightarrow{m_B} B$ in Definition 2.18 (2) is a coreflection sequence for B .

Proof. (a) follows from Corollary 2.13. Dually for (b). \square

Lemma 2.20. For any $B \in \mathcal{B}$, the following are equivalent.

- (a) $B \in \mathcal{U}$.
- (b) $B^+ \in \mathcal{W}$. Namely, it is isomorphic to 0 in $\underline{\mathcal{B}}^+$.
- (c) $\underline{p}_B = 0$ in $\underline{\mathcal{B}}$.

Proof. (a) implies $\underline{\mathcal{B}}^+(B^+, \underline{\mathcal{B}}^+) \cong \underline{\mathcal{B}}(B, \underline{\mathcal{B}}^+) = 0$ by Remark 2.7, which shows (b).

(b) \Rightarrow (c) is obvious. (c) \Rightarrow (a) follows from Lemma 2.17. Indeed, $\underline{p}_B = 0$ implies that id_B factors through some object in \mathcal{U} , which means $B \in \mathcal{U}$ by Remark 2.2. \square

Definition 2.21.

- (1) By Corollary 2.16 and Proposition 2.19, the inclusion functor $i^+ : \underline{\mathcal{B}} \hookrightarrow \underline{\mathcal{B}}^+$ has a right adjoint functor σ^+ . In particular, this σ^+ can be given by the following.
 - For each $B \in \mathcal{B}$, choose a reflection sequence $B \xrightarrow{p_B} B^+ \twoheadrightarrow S^U$ as in Definition 2.18 (1), and put $\sigma^+(B) = B^+$. If B itself belongs to \mathcal{B}^+ , we may choose as $B^+ = B$.
 - For any morphism $\underline{f} : B \rightarrow C$, we define $\sigma^+(\underline{f})$ as the unique morphism which makes the following diagram commutative.

$$\begin{array}{ccc} B & \xrightarrow{\underline{f}} & C \\ \underline{p}_B \downarrow & & \downarrow \underline{p}_C \\ B^+ & \xrightarrow[\sigma^+(\underline{f})]{} & C^+ \end{array}$$

- (2) Dually, the inclusion $i^- : \underline{\mathcal{B}}^- \hookrightarrow \underline{\mathcal{B}}$ has a left adjoint functor σ^- , defined by using a coreflection sequence as in Definition 2.18 (2).

Proposition 2.22. The functor σ^+ has the following properties.

- (a) σ^+ is an additive functor.
- (b) $\sigma^+|_{\underline{\mathcal{B}}^+} = \text{id}_{\underline{\mathcal{B}}^+}$.
- (c) For any morphism $f : A \rightarrow B$, $\sigma^+(\underline{f}) = 0$ holds in $\underline{\mathcal{B}}$ if and only if f factors through \mathcal{U} .

Dually for σ^- .

Proof. (a),(b) can be concluded easily by definition. By the adjoint property, $\sigma^+(\underline{f}) = 0$ is equivalent to $\underline{p_B f} = \sigma^+(\underline{f}) \circ \underline{p_A} = 0$. Thus (c) follows from Lemma 2.17. \square

Lemma 2.23. *Let*

$$\begin{array}{ccccc} A & \xrightarrow{w} & W & \longrightarrow & S \xrightarrow{\rho} \gg \\ f \downarrow & & \downarrow g & & \parallel \\ B & \xrightarrow{c} & C & \longrightarrow & S \xrightarrow{f_* \rho} \gg \end{array} \quad (2.4)$$

be any morphism of \mathbb{E} -triangles satisfying $W \in \mathcal{W}$ and $S \in \mathcal{S}$. Then we have the following.

(1) For any $X \in \mathcal{B}$,

$$\mathcal{B}(C, X) \xrightarrow{-\circ c} \mathcal{B}(B, X) \xrightarrow{-\circ f} \underline{\mathcal{B}}(A, X)$$

is exact.

(2) For any $Y \in \mathcal{B}^+$,

$$0 \rightarrow \underline{\mathcal{B}}(C, Y) \xrightarrow{-\circ c} \underline{\mathcal{B}}(B, Y) \xrightarrow{-\circ f} \underline{\mathcal{B}}(A, Y)$$

is exact.

(3) $\sigma^+(A) \xrightarrow{\sigma^+(c)} \sigma^+(B) \xrightarrow{\sigma^+(f)} \sigma^+(C) \rightarrow 0$ is a cokernel sequence in $\underline{\mathcal{B}}^+$.

Proof. (1) Suppose that a morphism $x \in \mathcal{B}(B, X)$ satisfies $\underline{x f} = 0$. This means that $x f$ factors through an object in \mathcal{W} . Thus $(x f)_* \rho = 0$ follows from $\mathbb{E}(S, \mathcal{W}) = 0$. By the exactness of

$$\mathcal{B}(C, X) \xrightarrow{-\circ c} \mathcal{B}(B, X) \xrightarrow{(f_* \rho)^\sharp} \mathbb{E}(S, X),$$

we obtain a morphism $x' \in \mathcal{B}(C, X)$ satisfying $x' c = x$.

(2) It suffices to show that $\underline{\mathcal{B}}(C, Y) \xrightarrow{-\circ c} \underline{\mathcal{B}}(B, Y)$ is monomorphic. Let $y \in \mathcal{B}(C, Y)$ be any morphism satisfying $\underline{y c} = 0$. Then there is a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{c} & C \\ w_1 \downarrow & & \downarrow y \\ W_0 & \xrightarrow{w_2} & Y \end{array}$$

for some object $W_0 \in \mathcal{W}$. By the exactness of $\mathcal{B}(C, W_0) \xrightarrow{c^*} \mathcal{B}(B, W_0) \rightarrow 0$, there is $w_3 \in \mathcal{B}(C, W_0)$ which gives $w_1 = w_3 c$. Then $y - w_2 w_3 \in \mathcal{B}(C, Y)$ satisfies $(y - w_2 w_3) c = 0$, and thus it factors through S . Since $\underline{\mathcal{B}}(S, Y) = 0$, it follows that $\underline{y} = \underline{y - w_2 w_3} = 0$.

(3) Indeed, by the adjoint property of σ^+ and (2),

$$0 \rightarrow \underline{\mathcal{B}}^+(\sigma^+(C), Y) \xrightarrow{-\circ \sigma^+(c)} \underline{\mathcal{B}}^+(\sigma^+(B), Y) \xrightarrow{-\circ \sigma^+(f)} \underline{\mathcal{B}}^+(\sigma^+(A), Y)$$

becomes exact for any $Y \in \mathcal{B}^+$. \square

Remark that the existence of a diagram (2.4) implies $A \in \mathcal{B}^-$. Conversely, for any morphism $f: A \rightarrow B$ from $A \in \mathcal{B}^-$, we can construct such a diagram in the following way.

Definition 2.24. For any morphism $f \in \mathcal{B}(A, B)$ with $A \in \mathcal{B}^-$, define $C_f \in \mathcal{B}$ and $c_f \in \mathcal{B}(B, C_f)$ as follows.

- Take an \mathbb{E} -triangle $A \xrightarrow{w^A} W^A \rightarrow S^A \xrightarrow{\rho^A} \gg$ and realize $f_* \rho^A \in \mathbb{E}(S^A, B)$ by an \mathbb{E} -triangle $B \xrightarrow{c_f} C_f \xrightarrow{s} S^A \xrightarrow{f_* \rho^A} \gg$.

This gives a morphism of \mathbb{E} -triangles, as follows. Remark that C_f is unique up to isomorphism in \mathcal{B} .

$$\begin{array}{ccccccc}
 A & \xrightarrow{w^A} & W^A & \longrightarrow & S^A & \xrightarrow{\rho^A} & \gg \\
 \downarrow f & & \downarrow & & \parallel & & \\
 B & \xrightarrow{c_f} & C_f & \xrightarrow{s} & S^A & \xrightarrow{f_* \rho^A} & \gg
 \end{array} \tag{2.5}$$

Moreover if $B \in \mathcal{B}^-$, then $C_f \in \mathcal{B}^-$ holds by Lemma 2.9.

2.2. The heart is semi-abelian.

Proposition 2.25. *For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, its heart $\underline{\mathcal{H}}$ is preabelian. Namely, any morphism in $\underline{\mathcal{H}}$ has a kernel and a cokernel.*

Proof. We only show how to construct the cokernel. Dual construction gives the kernel. Let $f \in \mathcal{H}(A, B)$ be any morphism. Take C_f and c_f as in Definition 2.24. Since $B \in \mathcal{H}$, it follows $C_f \in \mathcal{B}^-$. By Proposition 2.19, there exists $p = p_{C_f}: C_f \rightarrow C_f^+$ with $C_f^+ \in \mathcal{H}$. Lemma 2.23 (3) shows that $A \xrightarrow{f} B \xrightarrow{pc_f} C_f^+ \rightarrow 0$ is a cokernel sequence in $\underline{\mathcal{B}}^+$. Since all A, B, C_f^+ belong to $\underline{\mathcal{H}}$, this gives a cokernel sequence in $\underline{\mathcal{H}}$. \square

Corollary 2.26. *For any morphism $f: A \rightarrow B$ in \mathcal{H} , the following are equivalent.*

- (a) \underline{f} is an epimorphism in $\underline{\mathcal{H}}$.
- (b) $C_f^+ \in \mathcal{W}$.
- (c) $C_f \in \mathcal{U}$.

Proof. The equivalence of (b) and (c) is given by Lemma 2.20.

By Proposition 2.25, $\underline{pc_f}$ is the cokernel of \underline{f} in $\underline{\mathcal{H}}$. The equivalence of (a) and (b) follows immediately by this argument. \square

Definition 2.27. (cf. [R, Proposition 1]) *A preabelian category \mathcal{A} is called left semi-abelian if in any pull-back diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{\mathbf{a}} & B \\
 \downarrow \mathbf{b} & & \downarrow \mathbf{c} \\
 C & \xrightarrow{\mathbf{d}} & D
 \end{array}$$

in \mathcal{A} , the morphism \mathbf{a} is an epimorphism whenever \mathbf{d} is a cokernel morphism. Dually for a right semi-abelian category. \mathcal{A} is called semi-abelian if it is both left and right semi-abelian.

In this section we will prove that the heart $\underline{\mathcal{H}}$ of a twin cotorsion pair is semi-abelian.

Lemma 2.28. *If morphism $\mathbf{b} \in \underline{\mathcal{H}}(B, C)$ is a cokernel of a morphism $\underline{f} \in \underline{\mathcal{H}}(A, B)$, then there is a conflation $B \xrightarrow{b'} C' \rightarrow S$ with $S \in \mathcal{S}$ and $C' \in \mathcal{H}$, which admits an isomorphism $\mathbf{c} \in \underline{\mathcal{H}}(C, C')$ satisfying $\underline{b'} = \mathbf{c}\mathbf{b}$.*

Proof. For a morphism $f \in \mathcal{H}(A, B)$, the cokernel of \underline{f} is given by $\underline{pc_f}$, as in the proof of Proposition 2.25. Therefore, replacing by an isomorphism in $\underline{\mathcal{H}}$, we may assume $C = C_f$ and $\mathbf{b} = \underline{pc_f}$ from the beginning. Then by its definition, we have a morphism of \mathbb{E} -triangles (2.5), and a reflection sequence $C_f \rightarrow C_f^+ \rightarrow S'$

with $S' \in \mathcal{S}$. By (ET4), we obtain the following commutative diagram made of conflations.

$$\begin{array}{ccccc} B & \xrightarrow{c_f} & C_f & \twoheadrightarrow & S^A \\ \parallel & & \downarrow p & & \downarrow \\ B & \twoheadrightarrow & C_f^+ & \twoheadrightarrow & \exists S \\ & & \downarrow & & \downarrow \\ & & S' & \equiv & S' \end{array}$$

The extension-closedness of $\mathcal{S} \subseteq \mathcal{B}$ shows $S \in \mathcal{S}$. \square

Proposition 2.29. *Let $f \in \mathcal{H}(A, B)$ be any morphism. If $f^*: \mathbb{E}(B, V) \rightarrow \mathbb{E}(A, V)$ is monomorphic for any $V \in \mathcal{V}$, then \underline{f} is an epimorphism in $\underline{\mathcal{H}}$.*

Proof. Let (2.5) be a morphism of \mathbb{E} -triangles, as in Definition 2.24. Resolve C_f by an \mathbb{E} -triangle

$$V \rightarrow U \xrightarrow{u} C_f \xrightarrow{\lambda} 0 \quad (U \in \mathcal{U}, V \in \mathcal{V}).$$

From (2.5), we obtain a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{E}(C_f, V) \xrightarrow{c_f^*} \mathbb{E}(B, V) \\ & & \downarrow \quad \downarrow f^* \\ 0 & \longrightarrow & \mathbb{E}(A, V). \end{array}$$

where the top row is exact. Since f^* is monomorphic, it follows that $c_f^* = 0: \mathbb{E}(C_f, V) \rightarrow \mathbb{E}(B, V)$.

Thus by the exactness of

$$\mathcal{B}(B, U) \xrightarrow{u \circ \bar{}} \mathcal{B}(B, C_f) \xrightarrow{\lambda_\#} \mathbb{E}(B, V)$$

and $\lambda_\#(c_f) = c_f^* \lambda = 0$, there exists $g \in \mathcal{B}(B, U)$ satisfying $c_f = ug$. Since $\underline{\mathcal{B}}(U, C_f^+) = 0$, it follows that $\underline{pc}_f = \underline{pug} = 0$, which means $\text{Cok } \underline{f} \cong 0$ in $\underline{\mathcal{H}}$. \square

Lemma 2.30. *Suppose that $X \in \mathcal{B}^-$ admits a conflation*

$$X \xrightarrow{x} B \twoheadrightarrow U$$

with $B \in \mathcal{H}$ and $U \in \mathcal{U}$. Then $\sigma^+(\underline{x}) \in \underline{\mathcal{H}}(X^+, B)$ is an epimorphism in $\underline{\mathcal{H}}$.

Proof. Let

$$X \xrightarrow{p_X} X^+ \rightarrow S \dashrightarrow \quad (S \in \mathcal{S})$$

be an \mathbb{E} -triangle which gives a reflection sequence. By Proposition 2.15, there exists $y: X^+ \rightarrow B$ satisfying $yp_X = x$. This gives $\sigma^+(\underline{x}) = \underline{y}$. For any $V \in \mathcal{V}$, from the morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} X & \xrightarrow{p_X} & X^+ & \longrightarrow & S \dashrightarrow \\ \parallel & & \downarrow y & & \downarrow \exists \\ X & \xrightarrow{x} & B & \longrightarrow & U \dashrightarrow \end{array}$$

we obtain the following commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{E}(B, V) \xrightarrow{x^*} \mathbb{E}(X, V) \\ & & \downarrow y^* \quad \parallel \\ 0 & \longrightarrow & \mathbb{E}(X^+, V) \xrightarrow{p_X^*} \mathbb{E}(X, V) \end{array}$$

in which rows are exact. Thus $y^*: \mathbb{E}(B, V) \rightarrow \mathbb{E}(X^+, V)$ is monomorphic. By Proposition 2.29, this means that \underline{y} is epimorphic in $\underline{\mathcal{H}}$. \square

Lemma 2.31. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{a}} & B \\ \mathbf{b} \downarrow & & \downarrow \mathbf{c} \\ C & \xrightarrow{\mathbf{d}} & D \end{array} \quad (2.6)$$

be a pull-back diagram in $\underline{\mathcal{H}}$. If there exists an object $X \in \mathcal{B}^-$ and morphisms $x_B: X \rightarrow B$, $x_C: X \rightarrow C$ which satisfy the following conditions, then \mathbf{a} is an epimorphism in $\underline{\mathcal{H}}$.

(a) *The following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{x_B} & B \\ \underline{x_C} \downarrow & & \downarrow \mathbf{c} \\ C & \xrightarrow{\mathbf{d}} & D \end{array} \quad (2.7)$$

(b) *There exists a conflation $X \rightrightarrows B \twoheadrightarrow U$ with $U \in \mathcal{U}$.*

Proof. By the adjoint property of σ^+ , (2.7) induces a commutative diagram

$$\begin{array}{ccc} X^+ & \xrightarrow{\sigma^+(x_B)} & B \\ \sigma^+(\underline{x_C}) \downarrow & & \downarrow \mathbf{c} \\ C & \xrightarrow{\mathbf{d}} & D \end{array}$$

in $\underline{\mathcal{H}}$. Since (2.6) is a pull-back, there exists a morphism $\mathbf{e}: X^+ \rightarrow A$ in $\underline{\mathcal{H}}$ which makes the following diagram commute.

$$\begin{array}{ccccc} & & X^+ & & \\ & & \searrow \mathbf{e} & \searrow \sigma^+(x_B) & \\ & & A & \xrightarrow{\mathbf{a}} & B \\ & \searrow \sigma^+(\underline{x_C}) & \downarrow \mathbf{b} & & \downarrow \mathbf{c} \\ & & C & \xrightarrow{\mathbf{d}} & D \end{array}$$

Since $\sigma^+(\underline{x_B})$ is epimorphic by Lemma 2.30, it follows that \mathbf{a} is also an epimorphism. \square

Theorem 2.32. *For any twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$, its heart $\underline{\mathcal{H}}$ is semi-abelian.*

Proof. By duality, we only show that $\underline{\mathcal{H}}$ is left semi-abelian. Assume we are given a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{a}} & B \\ \mathbf{b} \downarrow & & \downarrow \mathbf{c} \\ C & \xrightarrow{\mathbf{d}} & D \end{array}$$

in $\underline{\mathcal{H}}$ where \mathbf{d} is a cokernel. Let us show that \mathbf{a} is an epimorphism.

By Lemma 2.28, replacing D by an isomorphism in $\underline{\mathcal{H}}$ if necessary, we can assume that there exists an inflation $d: C \rightarrow D$ satisfying $\mathbf{d} = \underline{d}$, which admits a conflation

$$C \rightrightarrows D \twoheadrightarrow S$$

with $S \in \mathcal{S}$. Since $D \in \mathcal{H}$, replacing B by an isomorphism as in Lemma 2.8 (1), we may assume that there exists conflation $B' \rightarrowtail B \xrightarrow{c} D$ such that $\mathbf{c} = \underline{c}$. By (ET4)^{op}, we get the following commutative diagram made of conflations.

$$\begin{array}{ccccc}
 B' & \xlongequal{\quad} & B' & & \\
 \downarrow & & \downarrow & & \\
 \exists X & \xrightarrow{\exists x_B} & B & \twoheadrightarrow & S \\
 \downarrow \exists x_C & & \downarrow c & & \parallel \\
 C & \xrightarrow{d} & D & \twoheadrightarrow & S
 \end{array}$$

Then $X \in \mathcal{B}^-$ follows from Lemma 2.9. Hence \mathbf{a} is epimorphic in $\underline{\mathcal{H}}$ by Lemma 2.31. \square

2.3. Functor to the heart.

Proposition 2.33. *There exists an isomorphism of functors from $\underline{\mathcal{B}}$ to $\underline{\mathcal{H}}$*

$$\eta: \sigma^+ \circ \sigma^- \xrightarrow{\cong} \sigma^- \circ \sigma^+.$$

Proof. Let σ^+, σ^- be the functors defined in Definition 2.21, by using \mathbb{E} -triangles

$$B \xrightarrow{p_B} B^+ \rightarrow S \xrightarrow{\delta} \quad \text{and} \quad V \rightarrow B^- \xrightarrow{m_B} B \xrightarrow{\mu} \quad (S \in \mathcal{S}, V \in \mathcal{V})$$

as in Definition 2.18. Remark that $\{p_B\}_{B \in \underline{\mathcal{B}}}$ forms a natural transformation $\text{id}_{\underline{\mathcal{B}}} \Rightarrow i^+ \circ \sigma^+$, which is the unit for the adjoint pair $i^+ \dashv \sigma^+$. Dually, $\{m_B\}_{B \in \underline{\mathcal{B}}}$: $i^- \circ \sigma^- \Rightarrow \text{id}_{\underline{\mathcal{B}}}$ gives the counit of the adjoint pair $\sigma^- \dashv i^-$.

By these adjointness, it can be easily shown that $\{p_B m_B\}_{B \in \underline{\mathcal{B}}}$: $i^- \circ \sigma^- \Rightarrow i^+ \circ \sigma^+$ induces a natural transformation $\eta: \sigma^+ \sigma^- \Rightarrow \sigma^- \sigma^+$, in which $\eta_B \in \underline{\mathcal{H}}(\sigma^+ \sigma^-(B), \sigma^- \sigma^+(B))$ is determined by the commutativity of the following diagram, for each $B \in \underline{\mathcal{B}}$.

$$\begin{array}{ccc}
 B^- & \xrightarrow{p_B m_B} & B^+ \\
 p_{(B^-)} \downarrow & & \uparrow m_{(B^+)} \\
 \sigma^+(B^-) & \xrightarrow{\eta_B} & \sigma^-(B^+)
 \end{array}$$

Let us show that η_B is an isomorphism for any $B \in \underline{\mathcal{B}}$. By the uniqueness of (co)reflection, it suffices to show that there exist an object $Q \in \mathcal{H}$ and morphisms $q \in \mathcal{B}(Q, B^+), \ell \in \mathcal{B}(B^-, Q)$, which satisfies the following conditions.

- (Q, ℓ) is a reflection of B^- along i^+ ,
- (Q, q) is a coreflection of B^+ along i^-
- the diagram

$$\begin{array}{ccc}
 B^- & \xrightarrow{p_B m_B} & B^+ \\
 \ell \downarrow & & \uparrow q \\
 Q & \xlongequal{\quad} & Q
 \end{array}$$

is commutative.

Since $p_B^*: \mathbb{E}(B^+, V) \xrightarrow{p_B^*} \mathbb{E}(B, V)$ is an isomorphism, there is $\theta \in \mathbb{E}(B^+, V)$ which gives $p_B^* \theta = \mu$. Realize θ as $V \rightarrow Q \xrightarrow{q} B^+ \xrightarrow{\theta}$. Then by (ET4)^{op}, we obtain the following commutative diagram made

of \mathbb{E} -triangles

$$\begin{array}{ccccc}
 V & \xrightarrow{\quad} & V & & \\
 \downarrow & & \downarrow & & \\
 \exists K & \xrightarrow{\exists \ell} & Q & \xrightarrow{\quad} & S \xrightarrow{\exists \nu} \gg \\
 \downarrow \exists k & & \downarrow q & & \parallel \\
 B & \xrightarrow{p_B} & B^+ & \xrightarrow{\quad} & S \xrightarrow{\delta} \gg \\
 \downarrow p_B^* \theta & & \downarrow \theta & & \\
 \Psi & & \Psi & &
 \end{array} \tag{2.8}$$

satisfying $k_*\nu = \delta$. Since $p_B^*\theta = \mu$, there is an isomorphism $B^- \xrightarrow{\cong} K$ which makes

$$\begin{array}{ccccc}
 & & B^- & & \\
 & \nearrow & \downarrow \cong & \searrow m_B & \\
 V & & & & B \\
 & \searrow & \downarrow k & \nearrow & \\
 & & K & &
 \end{array}$$

commutative. Thus we may replace K and k by B^- and m_B from the first. By Lemmas 2.9 and 2.10 applied to (2.8), we have $Q \in \mathcal{H}$. Resolve B as

$$V_B \xrightarrow{v_B} U_B \xrightarrow{u_B} B \xrightarrow{\lambda_B} \quad (U_B \in \mathcal{U}, V_B \in \mathcal{V}).$$

Then by Proposition 2.12, there exists $\sigma \in \mathbb{E}(S, U_B)$ satisfying $(u_B)_*\sigma = \delta$.

By $\mathbb{E}(U_B, V) = 0$, there is $t \in \mathcal{B}(U_B, B^-)$ satisfying $m_B t = u_B$. Then by the exactness of

$$0 \rightarrow \mathbb{E}(S, B^-) \xrightarrow{(m_B)^*} \mathbb{E}(S, B)$$

and the equality $(m_B)_*(\nu - t_*\sigma) = (m_B)_*\nu - (u_B)_*\sigma = \delta - \delta = 0$, we obtain $\nu = t_*\sigma$. Then we have a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc}
 U_B & \xrightarrow{\quad} & \exists X & \xrightarrow{\quad} & S \xrightarrow{\sigma} \gg \\
 \downarrow t & & \downarrow & & \parallel \\
 B^- & \xrightarrow{\ell} & Q & \xrightarrow{\quad} & S \xrightarrow{\nu} \gg
 \end{array}$$

By Corollary 2.13, homomorphism $\ell_*: \mathbb{E}(Q, V_0) \rightarrow \mathbb{E}(B^-, V_0)$ becomes surjective for any $V_0 \in \mathcal{V}$, which means that $B^- \xrightarrow{\ell} Q \rightarrow S$ is a reflection sequence. Dually, we can show that $V \rightarrow Q \xrightarrow{q} B^+$ is a coreflection sequence. This completes the proof. \square

Definition 2.34. Let $H: \mathcal{B} \rightarrow \underline{\mathcal{H}}$ denote the functor $\sigma^- \circ \sigma^+ \circ \pi$, where $\pi: \mathcal{B} \rightarrow \underline{\mathcal{B}}$ is the canonical quotient functor. By Proposition 2.33, there is a natural isomorphism of functors

$$H = \sigma^- \circ \sigma^+ \circ \pi \cong \sigma^+ \circ \sigma^- \circ \pi.$$

The following are the properties of H which will be used in section 3.2.

Remark 2.35. By Lemma 2.20 and its dual, we have $H(\mathcal{U}) = H(\mathcal{T}) = 0$.

Proposition 2.36. Let

$$\begin{array}{ccccccc}
 V & \xrightarrow{f} & A & \xrightarrow{a} & B & \dashrightarrow & \\
 \parallel & & \downarrow g & & \downarrow b & & \\
 V & \xrightarrow{v} & U & \xrightarrow{u} & C & \dashrightarrow &
 \end{array} \tag{2.9}$$

be a morphism of \mathbb{E} -triangles. Then

$$0 \rightarrow H(A) \xrightarrow{H(a)} H(B) \xrightarrow{H(b)} H(C) \quad (2.10)$$

is a kernel sequence in $\underline{\mathcal{H}}$.

Proof. [Reduction to the case where $U \in \mathcal{W}$ and $C \in \mathcal{B}^+$] Resolve U by an \mathbb{E} -triangle

$$U \rightarrow W \rightarrow S \dashrightarrow \quad (S \in \mathcal{S}, W \in \mathcal{T} \cap \mathcal{U} = \mathcal{W}).$$

Then by (ET4), we have the following diagram made of \mathbb{E} -triangles.

$$\begin{array}{ccccccc} V & \xrightarrow{\quad} & U & \xrightarrow{\quad} & C & \dashrightarrow & \triangleright \\ \parallel & & \downarrow & & \downarrow c & & \\ V & \xrightarrow{\quad} & W & \xrightarrow{\quad} & \exists C' & \dashrightarrow & \triangleright \\ & & \downarrow & & \downarrow & & \\ & & S & \xlongequal{\quad} & S & & \\ & & \downarrow & & \downarrow & & \\ & & \Psi & & \Psi & & \end{array} \quad (2.11)$$

Then C' belongs to $\text{Cone}(\mathcal{V}, \mathcal{W}) = \mathcal{B}^+$. By Corollary 2.13, $C \xrightarrow{c} C' \rightarrow S$ is a reflection sequence. Especially, $H(c)$ becomes an isomorphism. Composing the upper half of (2.11) with (2.9), we obtain the following morphism of \mathbb{E} -triangles.

$$\begin{array}{ccccccc} V & \xrightarrow{f} & A & \xrightarrow{a} & B & \dashrightarrow & \triangleright \\ \parallel & & \downarrow & & \downarrow cb & & \\ V & \xrightarrow{\quad} & W & \xrightarrow{\quad} & C' & \dashrightarrow & \triangleright \end{array}$$

Since $H(c)$ is an isomorphism, we may replace (2.9) by the above morphism. By this replacement, we may assume $U \in \mathcal{W}$ and $C \in \mathcal{B}^+$ from the beginning.

[Reduction to the case where $U \in \mathcal{W}$ and $A, B, C \in \mathcal{B}^+$] By the previous step, we may assume $U \in \mathcal{W}$ and $C \in \mathcal{B}^+$. Take an \mathbb{E} -triangle

$$A \xrightarrow{p_A} A^+ \rightarrow S_A \dashrightarrow \quad (A^+ \in \mathcal{B}^+, S_A \in \mathcal{S})$$

which gives a reflection sequence $A \xrightarrow{p_A} A^+ \rightarrow S_A$. By (ET4), we obtain the following commutative diagram made of \mathbb{E} -triangles.

$$\begin{array}{ccccccc} V & \xrightarrow{f} & A & \xrightarrow{a} & B & \dashrightarrow & \triangleright \\ \parallel & & \downarrow & & \downarrow p & & \\ V & \xrightarrow{p_A f} & A^+ & \xrightarrow{\quad} & \exists B' & \dashrightarrow & \triangleright \\ & & \downarrow & & \downarrow \exists \theta & & \\ & & S_A & \xlongequal{\quad} & S_A & & \\ & & \downarrow & & \downarrow & & \\ & & \Psi & & \Psi & & \end{array}$$

Then Lemma 2.10 (2) shows $B' \in \mathcal{B}^+$. Thus by Corollary 2.13,

$$B \xrightarrow{p} B' \rightarrow S_A$$

becomes a reflection sequence. Since $U \in \mathcal{W} \subseteq \mathcal{B}^+$, the sequence $\mathcal{B}^+(A^+, U) \rightarrow \mathcal{B}(A, U) \rightarrow 0$ is exact, and thus there is a morphism $q \in \mathcal{B}^+(A^+, U)$ satisfying $g = qp_A$. By (ET3), we obtain a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc} V & \xrightarrow{p_A f} & A^+ & \xrightarrow{\quad} & B' \xrightarrow{\theta} \gg \\ \parallel & & \downarrow q & & \downarrow \exists_{b'} \\ V & \xrightarrow{v} & U & \xrightarrow{u} & C \dashrightarrow \gg \end{array}$$

Since $H(p_A)$ and $H(p)$ are isomorphisms, we can replace (2.9) by this diagram.

By the above arguments, we may assume $U \in \mathcal{W}$ and $A, B, C \in \mathcal{B}^+$ from the beginning. Then the dual of Lemma 2.23 (3) shows that

$$0 \rightarrow \sigma^-(A) \xrightarrow{\sigma^-(a)} \sigma^-(B) \xrightarrow{\sigma^-(b)} \sigma^-(C)$$

is a kernel sequence in $\underline{\mathcal{B}}^-$. Since $A, B, C \in \mathcal{B}^+$, this means that (2.10) is a kernel sequence in $\underline{\mathcal{H}}$. \square

Corollary 2.37. *Let $A \xrightarrow{a} B \twoheadrightarrow S$ be any conflation, with $S \in \mathcal{S}$. Then $H(a)$ is an epimorphism in $\underline{\mathcal{H}}$.*

Proof. Resolve A by an \mathbb{E} -triangle $A \xrightarrow{t^A} T^A \xrightarrow{s^A} S^A \dashrightarrow$ with $S^A \in \mathcal{S}, T^A \in \mathcal{T}$. By Fact 1.19, we obtain the following commutative diagram made of \mathbb{E} -trinagles.

$$\begin{array}{ccccc} A & \xrightarrow{t^A} & T^A & \xrightarrow{s^A} & S^A \dashrightarrow \gg \\ \downarrow a & & \downarrow & & \parallel \\ B & \xrightarrow{\quad} & \exists X & \xrightarrow{\quad} & S^A \dashrightarrow \gg \\ \downarrow & & \downarrow & & \\ S & \xlongequal{\quad} & S & & \\ \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \end{array}$$

Since $\mathbb{E}(S, T^A) = 0$, we see that $T^A \rightarrow X \rightarrow S$ splits. Thus $X \cong T^A \oplus S$, which implies $H(X) = 0$ by Remark 2.35. By the dual of Proposition 2.36 applied to the upper half of this diagram, $H(A) \xrightarrow{H(a)} H(B) \rightarrow 0$ becomes exact. \square

3. HEARTS OF COTORSION PAIRS

In the rest of this article, we deal with single cotorsion pairs. For the notions introduced in the previous section, we continue to use the same symbol, applied to the case $(\mathcal{S}, \mathcal{T}) = (\mathcal{U}, \mathcal{V})$. Thus for example, we use $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$, $\mathcal{B}^+ = \text{Cone}(\mathcal{V}, \mathcal{W})$, $\mathcal{B}^- = \text{CoCone}(\mathcal{W}, \mathcal{S})$, $\mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^-$ and $\underline{\mathcal{H}} = \mathcal{H}/\mathcal{W}$.

3.1. The heart is abelian. In this section we fix a cotorsion pair $(\mathcal{U}, \mathcal{V})$. We will prove that the heart $\underline{\mathcal{H}} = \mathcal{B}^+ \cap \mathcal{B}^- / \mathcal{U} \cap \mathcal{V}$ of a cotorsion pair is abelian.

Lemma 3.1. *Let $A, B \in \mathcal{H}$, and let*

$$C \xrightarrow{g} A \xrightarrow{f} B \tag{3.1}$$

be any conflation in \mathcal{B} . If \underline{f} is epimorphic in $\underline{\mathcal{H}}$, then C belongs to \mathcal{B}^- .

Proof. Let $C \xrightarrow{g} A \xrightarrow{f} B \dashrightarrow$ be an \mathbb{E} -triangle which gives (3.1). Resolve A by an \mathbb{E} -triangle $A \xrightarrow{w^A} W^A \rightarrow U^A \xrightarrow{\rho^A} A$. By (ET4), we get the following commutative diagram made of \mathbb{E} -triangles.

$$\begin{array}{ccccc}
 C & \xlongequal{\quad} & C & & \\
 g \downarrow & & \downarrow h & & \\
 A & \xrightarrow{w^A} & W^A & \twoheadrightarrow & U^A \\
 f \downarrow & & \downarrow & & \parallel \\
 B & \xrightarrow{c_f} & C_f & \twoheadrightarrow & U^A.
 \end{array} \tag{3.2}$$

The lower half gives (2.5) in Definition 2.24. Since \underline{f} is an epimorphism, we have $C_f \in \mathcal{U}$ by Corollary 2.26. Remark that we have assumed $(\mathcal{S}, \mathcal{T}) = (\mathcal{U}, \mathcal{V})$. The middle column shows $C \in \mathcal{B}^-$. \square

Theorem 3.2. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on \mathcal{B} , its heart $\underline{\mathcal{H}}$ is an abelian category.*

Proof. Since $\underline{\mathcal{H}}$ is preabelian, it remains to show the following.

- (a) If \underline{f} is epimorphic in $\underline{\mathcal{H}}$, then \underline{f} is a cokernel of some morphism in $\underline{\mathcal{H}}$.
- (b) If \underline{f} is monomorphic in $\underline{\mathcal{H}}$, then \underline{f} is a kernel of some morphism in $\underline{\mathcal{H}}$.

We only show (a), since (b) is its dual. Let $\underline{f}: A \rightarrow B$ be any epimorphism in $\underline{\mathcal{H}}$. By Lemma 2.8, we may assume that f is a deflation, and thus there is a conflation $C \xrightarrow{g} A \xrightarrow{f} B$. Then by Lemma 3.1, it follows $C \in \mathcal{B}^-$. Moreover by its proof, we have a commutative diagram (3.2) made of \mathbb{E} -triangles.

If we take a reflection sequence $C \xrightarrow{p_C} C^+ \rightarrow U$ with $U \in \mathcal{U} = \mathcal{S}$, we obtain $C^+ \in \mathcal{H}$. By Proposition 2.15, there exists $a \in \mathcal{H}(C^+, A)$ which satisfies $ap_C = g$. Let us show that

$$C^+ \xrightarrow{a} A \xrightarrow{\underline{f}} B \rightarrow 0$$

is a cokernel sequence in $\underline{\mathcal{H}}$. By the assumption that \underline{f} is epimorphic and by the adjoint property of σ^+ , it suffices to show that

$$\underline{\mathcal{H}}(B, Q) \xrightarrow{-\circ \underline{f}} \underline{\mathcal{H}}(A, Q) \xrightarrow{-\circ g} \underline{\mathcal{H}}(C, Q)$$

is exact for any $Q \in \mathcal{H}$.

Let $r \in \mathcal{H}(A, Q)$ be any morphism satisfying $rg = 0$. By definition rg factors through some object in \mathcal{W} . Since the epimorphicity of \underline{f} implies $C_f \in \mathcal{U}$ in (3.2) by Corollary 2.26, this means that rf factors through h , and thus there exists $w \in \mathcal{B}(W^A, Q)$ satisfying $rg = wh$. Then the exactness of

$$\mathcal{B}(B, Q) \xrightarrow{-\circ \underline{f}} \mathcal{B}(A, Q) \xrightarrow{-\circ g} \mathcal{B}(C, Q)$$

and the equality $(r - ww^A)g = wh - wh = 0$ show that there exists $s \in \mathcal{B}(B, Q)$ which gives $r - ww^A = sf$. Thus we obtain $\underline{r} = \underline{s}\underline{f}$. \square

3.2. Associated cohomological functor. As in the case of a triangulated category, we define as follows.

Definition 3.3. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category as before, and let \mathcal{A} be an abelian category. An additive functor $F: \mathcal{B} \rightarrow \mathcal{A}$ is said to be cohomological, if any conflation $A \xrightarrow{x} B \xrightarrow{y} C$ yields an exact sequence $F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)$ in \mathcal{A} .*

Remark 3.4. If \mathcal{B} is an exact category, this means that F is a half exact functor.

As a corollary of Proposition 2.36 and Corollary 2.37, we obtain the following for a single cotorsion pair.

Theorem 3.5. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$, the associated functor $H: \mathcal{B} \rightarrow \underline{\mathcal{H}}$ is cohomological.*

Proof. Let us show the exactness of

$$H(A) \xrightarrow{H(x)} H(B) \xrightarrow{H(b)} H(C) \quad (3.3)$$

in $\underline{\mathcal{H}}$, for any \mathbb{E} -triangle $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$. Resolve C by an \mathbb{E} -triangle $V \xrightarrow{v} U \xrightarrow{u} C \dashrightarrow$ with $U \in \mathcal{U}, V \in \mathcal{V}$. By Fact 1.19, we obtain a commutative diagram made of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccccc} & & V & \xlongequal{\quad} & V & & \\ & & \downarrow & & \downarrow v & & \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & U & \dashrightarrow & \\ \parallel & & \downarrow b & & \downarrow u & & \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C & \dashrightarrow & \\ & & \downarrow & & \downarrow & & \end{array}$$

Then Proposition 2.36 and Corollary 2.37 applied to the case $\mathcal{S} = \mathcal{U}$ shows that

$$H(B') \xrightarrow{H(b)} H(B) \xrightarrow{H(y)} H(C) \quad \text{and} \quad H(A) \xrightarrow{H(x')} H(B') \rightarrow 0$$

are exact. This shows the exactness of (3.3). \square

We have the following corollaries.

Corollary 3.6. *For any $A, B \in \mathcal{H}$ and $f \in \mathcal{H}(A, B)$, the following are equivalent.*

- (1) \underline{f} is epimorphic in $\underline{\mathcal{H}}$.
- (2) There exist a conflation $A \xrightarrow{f'} B' \dashrightarrow U$ in \mathcal{B} with $U \in \mathcal{U}, B' \in \mathcal{H}$, and a morphism $b \in \mathcal{H}(B, B')$ which gives isomorphism $\underline{b}: B \xrightarrow{\cong} B'$ in $\underline{\mathcal{H}}$ satisfying $\underline{b}f = \underline{f}'$.

Proof. Suppose that \underline{f} is an epimorphism in $\underline{\mathcal{H}}$. Then in diagram (2.5), we have $C_f \in \mathcal{U}$ by Corollary 2.26. By Proposition 1.20, we obtain a conflation of the form

$$A \xrightarrow{\begin{pmatrix} -f \\ w^A \end{pmatrix}} B \oplus W^A \longrightarrow C_f \quad (W^A \in \mathcal{W}, C_f \in \mathcal{U}).$$

Modifying this by an isomorphism, $A \xrightarrow{\begin{pmatrix} f \\ -w^A \end{pmatrix}} B \oplus W^A \longrightarrow C_f$ also becomes a conflation. The converse follows from Remark 2.35 and Theorem 3.5. \square

Corollary 3.7. *Let*

$$\begin{array}{ccccc} X & \longrightarrow & K & \longrightarrow & C \dashrightarrow \\ \downarrow z & & \downarrow & & \parallel \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow \end{array}$$

be any morphism of \mathbb{E} -triangles satisfying $H(K) = 0$. Then we obtain an exact sequence $H(X) \xrightarrow{H(z)} H(A) \xrightarrow{H(x)} H(B) \xrightarrow{H(y)} H(C)$ in $\underline{\mathcal{H}}$.

Proof. By Proposition 1.20, we have a conflation of the form $X \xrightarrow{\begin{pmatrix} z \\ * \end{pmatrix}} A \oplus K \xrightarrow{(x \ *)} B$, similarly as in the proof of the previous corollary. Thus the above exactness follows from Theorem 3.5. \square

Corollary 3.8. *For any object $B \in \mathcal{B}$, the following are equivalent.*

- (a) $H(B) = 0$.
- (b) $B \in \text{add}(\mathcal{U} * \mathcal{V})$.

In particular, $\text{add}(\mathcal{U} * \mathcal{V}) \subseteq \mathcal{B}$ is extension-closed.

Proof. (b) \Rightarrow (a) follows from Remark 2.35 and Theorem 3.5. Let us show the converse. By Proposition 1.20 there is a conflation

$$U_B \rightarrowtail B \oplus T^U \twoheadrightarrow B^+$$

in the notation of (2.2). By the dual of Proposition 2.22, we see that $H(B) = \sigma^- \circ \sigma^+(B) = 0$ implies $B^+ \in \mathcal{V}$, and thus $B \in \text{add}(\mathcal{U} * \mathcal{V})$. \square

Definition 3.9. Let \mathcal{B} and $(\mathcal{U}, \mathcal{V})$ be as above.

- (1) We put $\mathcal{K} = \text{add}(\mathcal{U} * \mathcal{V})$, and call it the kernel of $(\mathcal{U}, \mathcal{V})$. Here, add denotes the closure under taking direct summands in \mathcal{B} .
- (2) We put $\mathcal{C} = {}^{\perp_1}\mathcal{K} = \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$, and call it the coheart of $(\mathcal{U}, \mathcal{V})$. Here ${}^{\perp_1}\mathcal{U} \subseteq \mathcal{B}$ denotes the subcategory of \mathcal{B} , consisting of those $X \in \mathcal{B}$ which satisfies $\mathbb{E}(X, \mathcal{U}) = 0$.

3.3. Heart-equivalence. We continue to use the notation $\sigma^+, \sigma^-, H, \mathcal{H}, \underline{\mathcal{H}}$ for a cotorsion pair $(\mathcal{U}, \mathcal{V})$. In this subsection, for another cotorsion pair $(\mathcal{U}', \mathcal{V}')$, we denote the corresponding notions for it by $\sigma'^+, \sigma'^-, H', \mathcal{H}', \underline{\mathcal{H}'}$ to distinguish.

Definition 3.10. Let $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}', \mathcal{V}')$ be cotorsion pairs. (We do not require that $((\mathcal{U}, \mathcal{V}), (\mathcal{U}', \mathcal{V}'))$ is a twin cotorsion pair.) They are said to be heart-equivalent, if there is an equivalence $E: \underline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{H}'}$ which makes

$$\begin{array}{ccc} & \mathcal{B} & \\ H \swarrow & & \searrow H' \\ \underline{\mathcal{H}} & \xrightarrow{E} & \underline{\mathcal{H}'} \end{array} \quad (3.4)$$

commutative up to natural isomorphism.

Lemma 3.11. Let $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}', \mathcal{V}')$ be arbitrary cotorsion pairs. Assume that they satisfy $H'(\mathcal{U}) = 0$. Then any reflection sequence with respect to $(\mathcal{U}, \mathcal{V})$

$$X \xrightarrow{p} Z \twoheadrightarrow U \quad (U \in \mathcal{U}) \quad (3.5)$$

gives an isomorphism $H'(p): H'(X) \xrightarrow{\cong} H'(Z)$ in $\underline{\mathcal{H}'}$.

If we assume $H'(\mathcal{V}) = 0$, then the dual holds for coreflection sequences with respect to $(\mathcal{U}, \mathcal{V})$.

Proof. Let $X \xrightarrow{p} Z \rightarrow U \xrightarrow{\nu} \rightarrow$ be an \mathbb{E} -triangle which gives (3.5). Resolve X by an \mathbb{E} -triangle

$$V_X \xrightarrow{v_X} U_X \xrightarrow{u_X} X \xrightarrow{\lambda_X} \rightarrow$$

with $U_X \in \mathcal{U}, V_X \in \mathcal{V}$. By Proposition 2.12, there is $\mu \in \mathbb{E}(U, U_X)$ satisfying $(u_X)_*\mu = \nu$. If we realize μ as $U_X \rightarrow U_0 \rightarrow U \xrightarrow{\mu} \rightarrow$, then $U_0 \in \mathcal{U}$ follows from the extension-closedness of $\mathcal{U} \subseteq \mathcal{B}$. We have a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc} U_X & \longrightarrow & U_0 & \longrightarrow & U \xrightarrow{\mu} \rightarrow \\ u_X \downarrow & & \downarrow \exists & & \parallel \\ X & \xrightarrow{p} & Z & \longrightarrow & U \xrightarrow{\nu} \rightarrow \end{array}$$

By Corollary 3.7, we obtain an exact sequence

$$0 \rightarrow H'(X) \xrightarrow{H'(p)} H'(Z) \rightarrow 0$$

in $\underline{\mathcal{H}'}$, which means that $H'(p)$ is an isomorphism. \square

Proposition 3.12. Let $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}', \mathcal{V}')$ be arbitrary cotorsion pairs, and let \mathcal{K} and \mathcal{K}' be their kernels, respectively. The following are equivalent.

- (1) $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}', \mathcal{V}')$ are heart-equivalent.

- (2) The equalities $H'(\mathcal{U}) = H'(\mathcal{V}) = 0$ and $H(\mathcal{U}') = H(\mathcal{V}') = 0$ are satisfied.
 (3) $\mathcal{K} = \mathcal{K}'$ holds.

Proof. By Corollary 3.8, (2) is equivalent to $\mathcal{U}, \mathcal{V} \subseteq \mathcal{K}'$ and $\mathcal{U}', \mathcal{V}' \subseteq \mathcal{K}$. Thus (3) \Rightarrow (2) is obvious, and (2) \Rightarrow (3) follows from the extension-closedness of \mathcal{K} and \mathcal{K}' in \mathcal{B} . Let us show (1) \Leftrightarrow (2).

(1) \Rightarrow (2) This immediately follows from Remark 2.35 and the commutativity of (3.4).

(2) \Rightarrow (1) Remark that $H'(\mathcal{U}) = 0$ implies that the functor H' factors through $\mathcal{B}/(\mathcal{U} \cap \mathcal{V})$ to give a functor $\mathcal{B}/(\mathcal{U} \cap \mathcal{V}) \rightarrow \underline{\mathcal{H}}'$. Particularly, composing with the inclusion $\underline{\mathcal{H}} \hookrightarrow \mathcal{B}/(\mathcal{U} \cap \mathcal{V})$, we obtain a functor $E: \underline{\mathcal{H}} \rightarrow \underline{\mathcal{H}}'$ which satisfies $E(X) = H'(X)$ for any object $X \in \underline{\mathcal{H}}$. Similarly, H induces a functor $E': \underline{\mathcal{H}}' \rightarrow \underline{\mathcal{H}}$.

Let us show the commutativity of (3.4). For each object $B \in \mathcal{B}$, take a reflection sequence

$$B \xrightarrow{p_B} \sigma^+(B) \twoheadrightarrow U_B \quad (U_B \in \mathcal{U})$$

and a coreflection sequence

$$V_B \hookrightarrow H(B) \xrightarrow{m_B} \sigma^+(B) \quad (V_B \in \mathcal{V})$$

with respect to $(\mathcal{U}, \mathcal{V})$. Then for any $B, C \in \mathcal{B}$ and any $f \in \mathcal{B}(B, C)$, the functoriality of σ^+ and σ^- gives the following commutative diagram in $\mathcal{B}/(\mathcal{U} \cap \mathcal{V})$.

$$\begin{array}{ccccc} B & \xrightarrow{p_B} & \sigma^+(B) & \xleftarrow{m_B} & H(B) \\ \downarrow f & & \downarrow \sigma^+(f) & & \downarrow H(f) \\ C & \xrightarrow{p_C} & \sigma^+(C) & \xleftarrow{m_C} & H(C) \end{array}$$

By Lemma 3.11, we obtain a commutative diagram in $\underline{\mathcal{H}}'$

$$\begin{array}{ccccccc} H'(B) & \xrightarrow[H'(\cong)]{H'(p_B)} & H'(\sigma^+(B)) & \xleftarrow[H'(\cong)]{H'(m_B)} & H'(H(B)) = E(H(B)) & & \\ \downarrow H'(f) & & \downarrow & & \downarrow E(H(f)) & & \\ H'(C) & \xrightarrow[H'(\cong)]{H'(p_C)} & H'(\sigma^+(C)) & \xleftarrow[H'(\cong)]{H'(m_C)} & H'(H(C)) = E(H(C)) & & \end{array}$$

in which the horizontal arrows are isomorphisms. Thus if we define $\zeta_B \in \underline{\mathcal{H}}'(H'(B), E(H(B)))$ by

$$\zeta_B = (H'(m_B))^{-1} \circ H'(p_B): H'(B) \rightarrow E(H(B)),$$

then the above commutativity shows that $\zeta = \{\zeta_B\}_{B \in \mathcal{B}}$ gives a natural isomorphism $\zeta: H' \xrightarrow{\cong} E \circ H$. Similarly, we can show the existence of a natural isomorphism $\zeta': H \xrightarrow{\cong} E' \circ H'$.

Composing with the inclusion $\mathcal{H} \hookrightarrow \mathcal{B}$, we see that

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \swarrow \pi & & \searrow \pi & \\ \underline{\mathcal{H}} & \xrightarrow{E} & \underline{\mathcal{H}}' & \xrightarrow{E'} & \underline{\mathcal{H}} \end{array}$$

is commutative up to natural isomorphism, where π is the canonical quotient functor. Then $E' \circ E \cong \text{Id}$ follows immediately. Similarly, we obtain $E \circ E' \cong \text{Id}$. \square

Let us write as $(\mathcal{U}, \mathcal{V}) \leq (\mathcal{U}', \mathcal{V}')$ when $\mathcal{V} \subseteq \mathcal{V}'$ holds, as in [S, §2]. Then for a fixed cotorsion pair $(\mathcal{U}, \mathcal{V})$, Proposition 3.12 tells that the largest *possible* cotorsion pair which is heart-equivalent to $(\mathcal{U}, \mathcal{V})$ should be $(\mathcal{C}, \mathcal{K})$. While the pair $(\mathcal{C}, \mathcal{K})$ is not always a cotorsion pair, a necessary and sufficient condition for it to be a cotorsion pair can be given by the existence of enough projectives of certain type in $\underline{\mathcal{H}}$. We will deal with this in the next section (Theorem 4.10).

Remark 3.13. The following is obvious.

- (1) If $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair, it is rigid by definition.
- (2) If $(\mathcal{U}, \mathcal{V})$ is rigid from the first, then $(\mathcal{C}, \mathcal{K}) = (\mathcal{U}, \mathcal{V})$ holds. Especially, $(\mathcal{C}, \mathcal{K})$ is indeed a cotorsion pair in this case.

4. HEARTS WITH ENOUGH PROJECTIVES

In the following sections, we assume that \mathcal{B} has enough projectives. As before, the subcategory of projectives is denoted by $\mathcal{P} \subseteq \mathcal{B}$. In this section, we give a sufficient condition for the heart $\underline{\mathcal{H}}$ to have enough projectives (Theorem 4.10), in terms of the kernel and the coheart. Moreover, under this condition, the heart admits an equivalence $\underline{\mathcal{H}} \simeq \text{mod}(\mathcal{C}/\mathcal{P})$ (Proposition 4.15).

The existence of enough projectives gives the following criterion.

Proposition 4.1. *Assume \mathcal{B} has enough projectives, as above. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$ be full additive subcategories, closed under taking direct summands and isomorphisms. Suppose that \mathcal{U} satisfies the following conditions.*

- (i) $\mathcal{P} \subseteq \mathcal{U}$.
- (ii) $\mathcal{U} \subseteq \mathcal{B}$ is extension-closed.

Then, the following are equivalent.

- (1) $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on \mathcal{B} .
- (2) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{B} = \text{CoCone}(\mathcal{V}, \mathcal{U})$ holds.

Proof. It suffices to show $\mathcal{B} = \text{Cone}(\mathcal{V}, \mathcal{U})$, under the assumption of (2). Let $B \in \mathcal{B}$ be any object. Since \mathcal{B} has enough projectives, there is a conflation $X \rightarrowtail P \twoheadrightarrow B$ with $P \in \mathcal{P}$. Since $\mathcal{B} = \text{CoCone}(\mathcal{V}, \mathcal{U})$, this X has a conflation $X \rightarrowtail V' \twoheadrightarrow U'$ with $U' \in \mathcal{U}, V' \in \mathcal{V}$. By Fact 1.19, we obtain the following commutative diagram made of conflations.

$$\begin{array}{ccccc}
 X & \rightarrowtail & P & \twoheadrightarrow & B \\
 \downarrow & & \downarrow & & \parallel \\
 V' & \rightarrowtail & \exists M & \twoheadrightarrow & B \\
 \downarrow & & \downarrow & & \\
 U' & \xlongequal{\quad} & U' & &
 \end{array}$$

By the assumption of (i),(ii), it follows $M \in \mathcal{U}$. □

4.1. Condition for the heart to have enough projectives. In the rest of this section, we fix a single cotorsion pair $(\mathcal{U}, \mathcal{V})$ on \mathcal{B} .

Definition 4.2. *For any subcategory $\mathcal{B}_1 \subseteq \mathcal{B}$, we define as $\Omega\mathcal{B}_1 = \text{CoCone}(\mathcal{P}, \mathcal{B}_1)$. Namely, $\Omega\mathcal{B}_1$ is the subcategory of \mathcal{B} consisting of objects ΩB_1 such that there exists a conflation*

$$\Omega B_1 \rightarrowtail P \twoheadrightarrow B_1$$

with $P \in \mathcal{P}$ and $B_1 \in \mathcal{B}_1$.

If subcategory $\mathcal{B}_1 \subseteq \mathcal{B}$ contains \mathcal{P} , the above definition can be described by the following. In the rest of this section, we write the quotient of such \mathcal{B}_1 by \mathcal{P} as $\overline{\mathcal{B}}_1 = \mathcal{B}_1/\mathcal{P}$. This is a full subcategory of $\overline{\mathcal{B}}$. For any morphism $f \in \mathcal{B}(X, Y)$, its image in $\overline{\mathcal{B}}(X, Y)$ will be denoted by \overline{f} .

Proposition 4.3. *The following correspondence gives a functor $\Omega: \overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}}$.*

- (1) *For each object B , choose an \mathbb{E} -triangle*

$$X \rightarrow P \rightarrow B \dashrightarrow \tag{4.1}$$

with $P \in \mathcal{P}$, and put $\Omega B = X$.

- (2) Let $\bar{f} \in \bar{\mathcal{B}}(B, B')$ be any morphism. Since P is projective, any representative $f \in \mathcal{B}(B, B')$ of \bar{f} induces a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} \Omega B & \longrightarrow & P & \longrightarrow & B \dashrightarrow \\ \downarrow g & & \downarrow & & \downarrow f \\ \Omega B' & \longrightarrow & P' & \longrightarrow & B' \dashrightarrow \end{array}$$

where the rows are the \mathbb{E} -triangles chosen in (1). We put $\Omega \bar{f} = \bar{g}$. This gives a well-defined homomorphism $\bar{\mathcal{B}}(B, B') \rightarrow \bar{\mathcal{B}}(\Omega B, \Omega B')$.

Moreover, this functor is uniquely determined up to natural isomorphism, independently from the choice of \mathbb{E} -triangles (4.1). In particular for each $B \in \mathcal{B}$, object ΩB is unique up to isomorphism in $\bar{\mathcal{B}}$. Remark that the image $\Omega \bar{\mathcal{B}}_1$ of $\bar{\mathcal{B}}_1$ by this functor agrees with the quotient $\overline{\Omega \mathcal{B}}_1 = (\Omega \mathcal{B}_1)/\mathcal{P}$, where $\Omega \mathcal{B}_1$ is as in Definition 4.2.

Proof. This can be shown in the same way as in [Ha, §2.2] and [IY, Proposition 2.6]. \square

Remark 4.4. Since $H(\mathcal{P}) = 0$, the functor $H: \mathcal{B} \rightarrow \underline{\mathcal{H}}$ induces a functor $\bar{H}: \bar{\mathcal{B}} \rightarrow \underline{\mathcal{H}}$ in a natural way. Then we have a sequence of functors $\bar{\mathcal{B}} \xrightarrow{\Omega} \bar{\mathcal{B}} \xrightarrow{\bar{H}} \underline{\mathcal{H}}$. In particular, if $B, B' \in \mathcal{B}$ satisfies $B \cong B'$ in $\bar{\mathcal{B}}$, then naturally $H(B) \cong H(B')$ holds in $\underline{\mathcal{H}}$.

Proposition 4.5. Let $A \xrightarrow{x} B \xrightarrow{y} C$ be any conflation. Then for any choice of ΩC , there exists $z \in \mathcal{B}(\Omega C, A)$ which makes

$$H(\Omega C) \xrightarrow{H(z)} H(A) \xrightarrow{H(x)} H(B) \xrightarrow{H(y)} H(C) \quad (4.2)$$

exact in $\underline{\mathcal{H}}$.

Proof. Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow$ be an \mathbb{E} -triangle, and let $\Omega C \xrightarrow{p} P \rightarrow C \dashrightarrow$ be any \mathbb{E} -triangle satisfying $P \in \mathcal{P}$. Since P is projective, we obtain a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc} \Omega C & \longrightarrow & P & \longrightarrow & C \dashrightarrow \\ \downarrow z & & \downarrow & & \parallel \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow \end{array}$$

Thus Corollary 3.7 shows the exactness of (4.2). \square

Since $\mathcal{P} \subseteq \mathcal{U}$, we have $\Omega \mathcal{U} = \text{CoCone}(\mathcal{P}, \mathcal{U}) \subseteq \mathcal{B}^-$ by Lemma 2.9 applied to the case $\mathcal{S} = \mathcal{U}$. Especially $\Omega \mathcal{C} \subseteq \mathcal{B}^-$ holds for the coheart \mathcal{C} . In particular, any $X \in \Omega \mathcal{C}$ satisfies $H(X) \cong \sigma^+(X)$ in $\underline{\mathcal{H}}$ by Proposition 2.33. We define as follows.

Definition 4.6. We denote by $H(\Omega \mathcal{C}) \subseteq \underline{\mathcal{H}}$ the essential image of $\Omega \mathcal{C}$ by the functor H . As above, $X \in \mathcal{H}$ belongs to $H(\Omega \mathcal{C})$ if and only if there exist an object $X' \in \mathcal{H}$ with an isomorphism $X \cong X'$ in $\underline{\mathcal{H}}$ and a reflection sequence

$$\Omega C \rightarrow X' \twoheadrightarrow U \quad (4.3)$$

for some $C \in \mathcal{C}$ and $U \in \mathcal{U}$.

Lemma 4.7. $\Omega \mathcal{C} \subseteq \mathcal{B}$ has the following properties.

- (1) For any $\Omega C \in \Omega \mathcal{C}$ and $B \in \mathcal{B}$, a morphism $f \in \mathcal{B}(\Omega C, B)$ factors through an object in \mathcal{P} if and only if it factors through one in \mathcal{U} .
- (2) $\bar{\mathcal{B}}(\Omega \mathcal{C}, \bar{\mathcal{U}}) = 0$.
- (3) $(\Omega \mathcal{C}) \cap \mathcal{U} = \mathcal{P}$.
- (4) For any $\Omega C \in \Omega \mathcal{C}$ and any conflation $A \xrightarrow{f} B \xrightarrow{g} U$ with $U \in \mathcal{U}$,

$$\bar{\mathcal{B}}(\Omega C, A) \xrightarrow{\bar{f} \circ -} \bar{\mathcal{B}}(\Omega C, B) \rightarrow 0$$

is exact.

Proof. (1) follows from the existence of a conflation

$$\Omega C \rightarrowtail P \twoheadrightarrow C \quad (C \in \mathcal{C}) \quad (4.4)$$

and $\mathbb{E}(C, \mathcal{U}) = 0$. (2) immediately follows from (1). (3) also follows from the existence of (4.4).

Let us show (4). Take any morphism $x \in \mathcal{B}(\Omega C, B)$. By (2) we have $\overline{g}x = 0$, and thus there exists $P \in \mathcal{P}$ and $p \in \mathcal{B}(\Omega C, P), q \in \mathcal{B}(P, U)$ satisfying $qp = gx$. Since P is projective, there is $r \in \mathcal{B}(P, B)$ which gives $q = gr$. Then, since $g \circ (x - rp) = 0$, we obtain $y \in \mathcal{B}(\Omega C, A)$ satisfying $fy = x - rp$. Thus it follows $\overline{x} = \overline{f}y$. \square

Proposition 4.8. *Any object $X \in H(\Omega C)$ is projective in $\underline{\mathcal{H}}$.*

Proof. Let us show the exactness of

$$\underline{\mathcal{H}}(X, A) \xrightarrow{f \circ -} \underline{\mathcal{H}}(X, B) \rightarrow 0 \quad (4.5)$$

for any epimorphism $\underline{f} \in \underline{\mathcal{H}}(A, B)$. Replacing X by an isomorphism in $\underline{\mathcal{H}}$, we may assume that there is a reflection sequence $\Omega C \xrightarrow{p} X \twoheadrightarrow U_X$ with $\Omega C \in \Omega \mathcal{C}$ and $U_X \in \mathcal{U}$. Similarly by Corollary 3.6, replacing B by an isomorphism, we may assume the existence of a conflation $A \xrightarrow{f} B \twoheadrightarrow U$ with $U \in \mathcal{U}$.

Let $x \in \underline{\mathcal{H}}(X, B)$ be any morphism. By Lemma 4.7 (4), we obtain $y \in \mathcal{B}(\Omega C, A)$ which makes

$$\begin{array}{ccc} \Omega C & \xrightarrow{\overline{p}} & X \\ \overline{g} \downarrow & & \downarrow \overline{x} \\ A & \xrightarrow{\overline{f}} & B \end{array}$$

commutative in $\overline{\mathcal{B}}$. Applying \overline{H} we obtain $H(\overline{f})H(y) = H(x)H(p)$, namely $\underline{f} \circ H(y) = \underline{x} \circ H(p)$ in $\underline{\mathcal{H}}$. Since $H(p)$ is an isomorphism, we obtain $\underline{x} = \underline{f} \circ (H(y)H(p)^{-1})$. This shows the exactness of (4.5). \square

Lemma 4.9. *For any $B \in \mathcal{B}$, the following are equivalent.*

- (1) *There exist $\Omega C \in \Omega \mathcal{C}$ and $d \in \mathcal{B}(\Omega C, B)$ which gives an epimorphism $H(d) \in \underline{\mathcal{H}}(H(\Omega C), H(B))$.*
- (2) *$B \in \text{CoCone}(\mathcal{K}, \mathcal{C})$.*

Proof. (2) \Rightarrow (1) follows from Corollary 3.8 and Proposition 4.5. Let us show the converse. Take an \mathbb{E} -triangle $\Omega C \rightarrow P \rightarrow C \xrightarrow{-\rho}$ with $P \in \mathcal{P}$ and $C \in \mathcal{C}$. Then $d \in \mathcal{B}(\Omega C, B)$ induces a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccc} \Omega C & \longrightarrow & P & \longrightarrow & C \xrightarrow{-\rho} \gg \\ d \downarrow & & \downarrow & & \parallel \\ B & \longrightarrow & \exists Y & \longrightarrow & C \xrightarrow[-d \circ \rho]{} \gg \end{array}$$

By Corollary 3.7, $H(\Omega C) \xrightarrow{H(d)} H(B) \rightarrow H(Y) \rightarrow 0$ becomes exact. Since $H(d)$ is an epimorphism, it follows $Y \in \mathcal{K}$. \square

Theorem 4.10. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$, the following are equivalent.*

- (1) *Any object $B \in \underline{\mathcal{H}}$ admits an epimorphism $X \rightarrow B$ in $\underline{\mathcal{H}}$ from some $X \in H(\Omega \mathcal{C})$. In particular by Proposition 4.8, $\underline{\mathcal{H}}$ has enough projectives.*
- (2) *$(\mathcal{C}, \mathcal{K})$ is a cotorsion pair on \mathcal{B} .*
- (3) *$\mathcal{B} = \text{CoCone}(\mathcal{K}, \mathcal{C})$.*
- (4) *$\mathcal{U} \subseteq \text{Cone}(\mathcal{K}, \mathcal{C})$.*

Moreover in this case, the subcategory of projectives in $\underline{\mathcal{H}}$ agrees with $\text{add}(H(\Omega \mathcal{C}))$. Here, add denotes the closure under taking direct summands in $\underline{\mathcal{H}}$.

Proof. The latter part is obvious, since any projective object come to have epimorphism from some object in $H(\Omega\mathcal{C})$, which necessarily splits. (3) \Rightarrow (1) follows immediately from Lemma 4.9 applied to each $B \in \mathcal{H}$. (2) \Leftrightarrow (3) follows from $\mathcal{C} = {}^{\perp_1}\mathcal{K}$ and Proposition 4.1. (2) \Rightarrow (4) is trivial.

(1) \Rightarrow (3) Let $B \in \mathcal{B}$ be any object. Take a reflection sequence

$$B \xrightarrow{p_B} B^+ \twoheadrightarrow U \quad (B^+ \in \mathcal{B}^+, U \in \mathcal{U})$$

and a coreflection sequence

$$V \hookrightarrow (B^+)^- \xrightarrow{m} B^+ \quad ((B^+)^- \in \mathcal{H}, V \in \mathcal{V}).$$

By assumption, there exist $X \in H(\Omega\mathcal{C})$ and an epimorphism $\underline{x}: X \rightarrow (B^+)^-$ in $\underline{\mathcal{H}}$. Replacing X by an isomorphism in $\underline{\mathcal{H}}$, we may assume the existence of a reflection sequence

$$\Omega\mathcal{C} \xrightarrow{p} X \twoheadrightarrow U \quad (U \in \mathcal{U}, C \in \mathcal{C}).$$

By Lemma 4.7 (4), there is a morphism $d \in \mathcal{B}(\Omega\mathcal{C}, B)$ which makes

$$\begin{array}{ccc} \Omega\mathcal{C} & \xrightarrow{\bar{p}} & X \\ \bar{d} \downarrow & & \downarrow \overline{mx} \\ B & \xrightarrow{\bar{p}_B} & B^+ \end{array}$$

commutative in $\overline{\mathcal{B}}$. Applying \overline{H} , we obtain $H(p_B)H(d) = H(m)H(x)H(p)$. Since $H(p_B), H(m), H(p)$ are isomorphisms and $H(x) = \underline{x}$ is an epimorphism, it follows that $H(d): H(\Omega\mathcal{C}) \rightarrow H(B)$ becomes an epimorphism. Thus Lemma 4.9 shows $B \in \text{CoCone}(\mathcal{K}, \mathcal{C})$.

(4) \Rightarrow (3) For any $B \in \mathcal{B}$, take a conflation $B \hookrightarrow V^B \twoheadrightarrow U^B$ with $U^B \in \mathcal{U}, V^B \in \mathcal{V}$. By assumption, there is a conflation $K \hookrightarrow C \twoheadrightarrow U^B$ with $K \in \mathcal{K}, C \in \mathcal{C}$. By Fact 1.19, we obtain a commutative diagram made of conflations as follows.

$$\begin{array}{ccccc} & & K & \xlongequal{\quad} & K \\ & & \downarrow & & \downarrow \\ B & \xrightarrow{\quad} & \exists M & \twoheadrightarrow & C \\ \parallel & & \downarrow & & \downarrow \\ B & \xrightarrow{\quad} & V^B & \twoheadrightarrow & U^B \end{array}$$

Theorem 3.5 and Corollary 3.8 shows $H(M) = 0$, and thus $M \in \mathcal{K}$. □

Corollary 4.11. *If $(\mathcal{U}, \mathcal{V})$ is a rigid cotorsion pair, then the heart $\underline{\mathcal{H}}$ has enough projectives.*

Proof. This immediately follows from Remark 3.13 (2) and Theorem 4.10. □

Remark 4.12. We may also show Corollary 4.11 directly, specializing the argument so far to rigid cotorsion pairs. Then in the way around, Corollary 4.11 and Proposition 3.12 will show that whenever $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair (which is necessarily rigid by Remark 3.13 (1)), the heart $\underline{\mathcal{H}}$ of $(\mathcal{U}, \mathcal{V})$ has enough projectives. We emphasize here that Theorem 4.10 is also giving its converse, a characterization for $(\mathcal{C}, \mathcal{K})$ to be a cotorsion pair.

4.2. Equivalence with the category of coherent functors. The following fact gives the desired equivalence between the heart and the functor category. For example, see [Be, Corollaries 3.9, 3.10] for the detail.

Fact 4.13. *If an abelian category \mathcal{A} has enough projectives, then there is an equivalence $\mathcal{A} \simeq \text{mod}(\text{Proj}\mathcal{A})$. Here, $\text{mod}(\text{Proj}\mathcal{A})$ denotes the category of coherent functors over the category of projectives $\text{Proj}\mathcal{A}$.*

Lemma 4.14. *The sequence of functors*

$$\overline{\mathcal{C}} \xrightarrow{\Omega} \Omega\overline{\mathcal{C}} \xrightarrow{\overline{H}} H(\Omega\mathcal{C}), \quad (4.6)$$

which is obtained by restricting $\overline{\mathcal{B}} \xrightarrow{\Omega} \overline{\mathcal{B}} \xrightarrow{\overline{H}} \underline{\mathcal{H}}$ in Remark 4.4 onto $\overline{\mathcal{C}}$, give equivalences of categories $\overline{\mathcal{C}} \xrightarrow{\sim} \Omega\overline{\mathcal{C}} \xrightarrow{\sim} H(\Omega\mathcal{C})$.

Proof. Both functors are essentially surjective by definition. It suffices to show that they are fully faithful.

1. **Faithfulness of $\overline{\mathcal{C}} \xrightarrow{\Omega} \Omega\overline{\mathcal{C}}$.** Suppose that $\overline{f} \in \overline{\mathcal{C}}(C, C')$ satisfies $\Omega\overline{f} = 0$. By definition, $\Omega\overline{f} = \overline{g}$ is given by a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} \Omega C & \xrightarrow{q} & P & \xrightarrow{p} & C \xrightarrow{\rho} \triangleright \\ g \downarrow & & \downarrow & & \downarrow f \\ \Omega C' & \xrightarrow{q'} & P' & \xrightarrow{p'} & C' \xrightarrow{\rho'} \triangleright \end{array} \quad (4.7)$$

where the two rows are \mathbb{E} -triangles satisfying $P, P' \in \mathcal{P}$. If $\overline{g} = 0$, then g factors through an projective object in \mathcal{P} . Since $\mathbb{E}(C, \mathcal{P}) = 0$, this implies $f^*\rho' = g_*\rho = 0$. Thus f factors through p' .

2. **Fullness of $\overline{\mathcal{C}} \xrightarrow{\Omega} \Omega\overline{\mathcal{C}}$.** Let $\overline{g} \in (\Omega\overline{\mathcal{C}})(\Omega C, \Omega C')$ be any morphism. Then by $\mathbb{E}(C, \mathcal{P}) = 0$, we obtain a morphism of \mathbb{E} -triangles as in (4.7), with some $f \in \mathcal{C}(C, C')$. This gives $\Omega\overline{f} = \overline{g}$.

3. **Faithfulness of $\Omega\overline{\mathcal{C}} \xrightarrow{\overline{H}} H(\Omega\mathcal{C})$.** This follows from Proposition 2.22 and Lemma 4.7 (1).

4. **Fullness of $\Omega\overline{\mathcal{C}} \xrightarrow{\overline{H}} H(\Omega\mathcal{C})$.** Let $X, X' \in H(\Omega\mathcal{C})$ be any pair of objects, and let $\underline{g} \in \underline{\mathcal{H}}(X, X')$ be any morphism. Replacing by isomorphisms in $\underline{\mathcal{H}}$, we may assume there exist conflations

$$\Omega C \xrightarrow{p} X \twoheadrightarrow C, \quad \Omega C' \xrightarrow{p'} X' \twoheadrightarrow C' \quad (C, C' \in \mathcal{C}).$$

By Lemma 4.7 (4), there is $f \in \mathcal{B}(\Omega C, \Omega C')$ which satisfies $\overline{p'}f = \overline{gp}$. Applying \overline{H} , we obtain $H(p')H(f) = H(g)H(p) = \underline{g} \circ H(p)$. This shows the fullness of $\overline{\mathcal{C}} \xrightarrow{\overline{H}} H(\Omega\mathcal{C})$. \square

Proposition 4.15. *If $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair, then the heart $\underline{\mathcal{H}}$ of $(\mathcal{U}, \mathcal{V})$ satisfies $\underline{\mathcal{H}} \simeq \text{mod}(\mathcal{C}/\mathcal{P})$.*

Proof. Since the inclusion $H(\Omega\mathcal{C}) \hookrightarrow H(\Omega\mathcal{C})$ induces an equivalence $\text{mod}(H(\Omega\mathcal{C})) \xrightarrow{\sim} \text{mod}(\text{add}(H(\Omega\mathcal{C})))$, this follows from Lemma 4.14, Theorem 4.10 and Fact 4.13. \square

5. RELATION WITH n -CLUSTER TILTING SUBCATEGORIES

In a triangulated category, if \mathcal{N} is a cluster tilting subcategory, then $(\mathcal{N}, \mathcal{N})$ is a cotorsion pair with the coheart \mathcal{N} and the heart \mathcal{T}/\mathcal{N} . We have an equivalence $\mathcal{T}/\mathcal{N} \simeq \text{mod}\mathcal{N} \simeq \text{mod}(\mathcal{N}[-1])$ (see [KZ, Corollary 4.4]). Fix an integer $n \geq 2$ throughout this section. We also have a similar result for any n -cluster tilting subcategory. If \mathcal{N} is an n -cluster tilting subcategory, then by [IY, Theorem 3.1], we have cotorsion pairs $(\mathcal{U}_\ell, \mathcal{V}_\ell) = (\mathcal{N} * \mathcal{N}[1] * \cdots * \mathcal{N}[\ell-1], \mathcal{N}[\ell-1] * \mathcal{N}[\ell] * \cdots * \mathcal{N}[n-2])$ for $0 < \ell < n$. Since the coheart of $(\mathcal{U}_\ell, \mathcal{V}_\ell)$ is $\mathcal{C}_\ell = \mathcal{N}$, its heart $\underline{\mathcal{H}}_\ell$ becomes equivalent to $\text{mod}\mathcal{N}$. In particular, the hearts of these cotorsion pairs $(\mathcal{U}_\ell, \mathcal{V}_\ell)$ are equivalent.

In this section we will generalize this to an extriangulated category with enough projectives and injectives. We define n -cluster tilting subcategory \mathcal{M} in \mathcal{B} and show how cotorsion pairs are induced from \mathcal{M} .

Assume that $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ has enough projectives and injectives, throughout this section.

5.1. Higher extensions. For a subcategory $\mathcal{B}_1 \subseteq \mathcal{B}$, put $\Omega^0 \mathcal{B}_1 = \mathcal{B}_1$, and define $\Omega^i \mathcal{B}_1$ for $i > 0$ inductively by

$$\Omega^i \mathcal{B}_1 = \Omega(\Omega^{i-1} \mathcal{B}_1) = \text{CoCone}(\mathcal{P}, \Omega^{i-1} \mathcal{B}_1).$$

We call $\Omega^i \mathcal{B}_1$ the i -th syzygy of \mathcal{B}_1 . Dually we define the i -th cosyzygy $\Sigma^i \mathcal{B}_1$ by $\Sigma^0 \mathcal{B}_1 = \mathcal{B}_1$ and $\Sigma^i \mathcal{B}_1 = \text{Cone}(\Sigma^{i-1} \mathcal{B}_1, \mathcal{I})$ for $i > 0$.

Let X be any object in \mathcal{B} . It admits an \mathbb{E} -triangle

$$X \rightarrow I^0 \rightarrow \Sigma X \xrightarrow{\delta^X} \quad (\text{resp. } \Omega X \rightarrow P_0 \rightarrow X \xrightarrow{\delta_X}),$$

where $I^0 \in \mathcal{I}$ (resp. $P_0 \in \mathcal{P}$). We can get \mathbb{E} -triangles

$$\Sigma^i X \rightarrow I^i \rightarrow \Sigma^{i+1} X \xrightarrow{\delta^{\Sigma^i X}} \quad (\text{resp. } \Omega^{i+1} \rightarrow P_i \rightarrow \Omega^i X \xrightarrow{\delta_{\Omega^i X}}),$$

for $i > 0$ recursively.

Lemma 5.1. *For any $A, X \in \mathcal{B}$ and \mathbb{E} -triangles*

$$A \rightarrow I^A \xrightarrow{i} \Sigma A \xrightarrow{\iota^A}, \quad \Omega X \xrightarrow{p} P \rightarrow X \xrightarrow{\rho}$$

satisfying $I^A \in \mathcal{I}$ and $P \in \mathcal{P}$, there is an isomorphism $\varphi_{X,A}: \mathbb{E}(\Omega X, A) \xrightarrow{\cong} \mathbb{E}(X, \Sigma A)$. Moreover if $B \rightarrow I^B \xrightarrow{i} \Sigma B \xrightarrow{\iota^B}$ is also an \mathbb{E} -triangle with $I^B \in \mathcal{I}$, then for any $a \in \mathcal{B}(A, B)$,

$$\begin{array}{ccc} \mathbb{E}(\Omega X, A) & \xrightarrow[\cong]{\varphi_{X,A}} & \mathbb{E}(X, \Sigma A) \\ a_* \downarrow & & \downarrow a_{1*} \\ \mathbb{E}(\Omega X, B) & \xrightarrow[\varphi_{X,B}]{\cong} & \mathbb{E}(X, \Sigma B) \end{array} \quad (5.1)$$

is commutative. Here, $a_1: \Sigma A \rightarrow \Sigma B$ is any morphism satisfying $a_1^* \iota^B = a_* \iota^A$. Remark that such a_1 exists by the injectivity of I^B (dually to (2) in Proposition 4.3).

Proof. We have the following exact sequences. Especially, $\iota_{\#}^A$ and $\rho_{\#}$ are surjective.

$$\begin{array}{ccccccc} \mathcal{B}(\Omega X, I^A) & \xrightarrow{i \circ -} & \mathcal{B}(\Omega X, \Sigma A) & \xrightarrow{\iota_{\#}^A} & \mathbb{E}(\Omega X, A) & \longrightarrow & 0 & \text{exact} \\ & & \parallel & & & & & \\ \mathcal{B}(P, \Sigma A) & \xrightarrow{- \circ p} & \mathcal{B}(\Omega X, \Sigma A) & \xrightarrow{\rho_{\#}} & \mathbb{E}(X, \Sigma A) & \longrightarrow & 0 & \text{exact} \end{array}$$

Since $I^A \in \mathcal{I}$, we have $\rho_{\#}(i \circ m) = i_* m_* \rho = 0$ for any $m \in \mathcal{B}(\Omega X, I^A)$. Thus there is a unique homomorphism $\varphi_{X,A}: \mathbb{E}(\Omega X, A) \rightarrow \mathbb{E}(X, \Sigma A)$ which satisfies $\rho_{\#} = \varphi_{X,A} \circ \iota_{\#}^A$. Similarly, we obtain a homomorphism $\psi_{X,A}: \mathbb{E}(X, \Sigma A) \rightarrow \mathbb{E}(\Omega X, A)$ satisfying $\psi_{X,A} \circ \rho_{\#} = \iota_{\#}^A$. By the uniqueness, this gives the inverse of $\varphi_{X,A}$.

Since we have

$$\begin{aligned} \varphi_{X,B} \circ a_* \circ \iota_{\#}^A(f) &= \varphi_{X,B}(f^* a_* \iota^A) = \varphi_{X,B}(f^* a_1^* \iota^B) \\ &= \varphi_{X,B} \circ \iota_{\#}^B(a_1 f) = \rho_{\#}(a_1 f) \\ &= a_{1*} \circ \rho_{\#}(f) = a_{1*} \circ \varphi_{X,A} \circ \iota_{\#}^A(f) \end{aligned}$$

for any $f \in \mathcal{B}(\Omega X, \Sigma A)$, commutativity of (5.1) follows from the surjectivity of $\iota_{\#}^A$. \square

Proposition 5.2. *Let $A \xrightarrow{a} B \xrightarrow{b} C$ be a conflation. For any object $X \in \mathcal{B}$, we have the following long exact sequence.*

$$\cdots \rightarrow \mathbb{E}(\Omega^i X, A) \xrightarrow{a_*} \mathbb{E}(\Omega^i X, B) \xrightarrow{b_*} \mathbb{E}(\Omega^i X, C) \rightarrow \mathbb{E}(\Omega^{i+1} X, A) \xrightarrow{a_*} \mathbb{E}(\Omega^{i+1} X, B) \xrightarrow{b_*} \cdots \quad (i > 0)$$

Proof. As in Lemma 5.1, take \mathbb{E} -triangles

$$A \xrightarrow{j} I^A \xrightarrow{i} \Sigma A \xrightarrow{\iota^A}, \quad B \rightarrow I^B \rightarrow \Sigma B \xrightarrow{\iota^B} \quad (I^A, I^B \in \mathcal{I})$$

and a morphism $a_1 \in \mathcal{B}(\Sigma A, \Sigma B)$ satisfying $a_1^* \iota^B = a_* \iota^A$. Since I^A is injective, any \mathbb{E} -triangle $A \xrightarrow{a} B \xrightarrow{b} C \dashrightarrow$ admits a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C & \dashrightarrow & \\ \parallel & & \downarrow k & & \downarrow c & & \\ A & \xrightarrow{j} & I^A & \xrightarrow{i} & \Sigma A & \dashrightarrow & \end{array}$$

We may modify k to induce an \mathbb{E} -triangle

$$B \xrightarrow{\begin{pmatrix} b \\ k \end{pmatrix}} C \oplus I^A \xrightarrow{(-c \ i)} \Sigma A \xrightarrow{a_* \iota^A}$$

by the dual of Proposition 1.20. Then $\mathbb{E}(X, B) \xrightarrow{b_*} \mathbb{E}(X, C) \xrightarrow{c_*} \mathbb{E}(X, \Sigma A)$ becomes exact. Since the equality $a_* \iota^A = a_1^* \iota^B$ induces a morphism of \mathbb{E} -triangles as follows,

$$\begin{array}{ccccccc} B & \xrightarrow{\begin{pmatrix} b \\ k \end{pmatrix}} & C \oplus I^A & \xrightarrow{(-c \ i)} & \Sigma A & \xrightarrow{a_* \iota^A} & \\ \parallel & & \downarrow & & \downarrow a_1 & & \\ B & \longrightarrow & I^B & \longrightarrow & \Sigma B & \xrightarrow{\iota^B} & \end{array}$$

the same argument shows that $\mathbb{E}(X, C) \xrightarrow{c_*} \mathbb{E}(X, \Sigma A) \xrightarrow{a_1^*} \mathbb{E}(X, \Sigma B)$ is exact. Hence we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccc} & & \mathbb{E}(\Omega X, A) & \xrightarrow{a_*} & \mathbb{E}(\Omega X, B) & \xrightarrow{b_*} & \mathbb{E}(\Omega X, C) \\ & & \downarrow \varphi_{X,A} \cong & & \downarrow \cong \varphi_{X,B} & & \\ \mathbb{E}(X, A) & \xrightarrow{a_*} & \mathbb{E}(X, B) & \xrightarrow{b_*} & \mathbb{E}(X, C) & \xrightarrow{-c_*} & \mathbb{E}(X, \Sigma A) \xrightarrow{a_1^*} \mathbb{E}(X, \Sigma B) \end{array}$$

Replacing X by $\Omega^i X$ recursively, we get the general case. \square

5.2. Cotorsion pairs induced from n -cluster tilting subcategories. For convenience, we denote $\mathbb{E}(X, \Sigma^i Y) \cong \mathbb{E}(\Omega^i X, Y)$ by $\mathbb{E}^{i+1}(X, Y)$ for $i \geq 0$ in the rest.

Definition 5.3. A full subcategory $\mathcal{M} \subseteq \mathcal{B}$ is called n -cluster tilting, if it satisfies the following conditions.

- (1) \mathcal{M} is contravariantly finite and covariantly finite in \mathcal{B} ,
- (2) $X \in \mathcal{M}$ if and only if $\mathbb{E}^i(X, \mathcal{M}) = 0$ for any $i \in \{1, 2, \dots, n-1\}$,
- (3) $X \in \mathcal{M}$ if and only if $\mathbb{E}^i(\mathcal{M}, X) = 0$ for any $i \in \{1, 2, \dots, n-1\}$.

In particular, $\mathcal{M} \subseteq \mathcal{B}$ becomes an additive subcategory closed by isomorphisms and direct summands, which contains all the projectives and all the injectives.

In the remaining of this article, let \mathcal{M} be an n -cluster tilting subcategory in \mathcal{B} .

Definition 5.4. For any $\ell \geq 0$, we define a full subcategory $\mathcal{M}_\ell \subseteq \mathcal{B}$ inductively as follows.

- $\mathcal{M}_1 = \mathcal{M}$.
- $\mathcal{M}_\ell = \text{Cone}(\mathcal{M}_{\ell-1}, \mathcal{M})$ for $\ell > 1$.

Definition 5.5. For any $m > 0$, denote by $\mathcal{M}^{\perp m}$ the subcategory of objects $X \in \mathcal{B}$ satisfying

$$\mathbb{E}^i(\mathcal{M}, X) = 0 \quad (1 \leq i \leq m).$$

We have $\mathcal{M}^{\perp_{n-1}} = \mathcal{M}$.

Lemma 5.6. *We have the following.*

$$\mathcal{M}_\ell = \begin{cases} \mathcal{M}^{\perp_{n-\ell}} & 0 \leq \ell < n, \\ \mathcal{B} & \ell \geq n. \end{cases}$$

In particular, $\mathcal{M}_\ell \subseteq \mathcal{B}$ is closed under direct summands. Moreover, we have $\mathcal{M}_k \subseteq \mathcal{M}_\ell$ for any $0 \leq k \leq \ell$.

Proof. This follows from Proposition 5.2. \square

Definition 5.7. For any $1 \leq j \leq i$, put $\mathcal{Y}_{i,j} = \Sigma^{j-1}\mathcal{M}_{i-j}$. In particular, for any $1 \leq \ell \leq n$, we put $\mathcal{Y}_\ell = \mathcal{Y}_{n,\ell} = \Sigma^{\ell-1}\mathcal{M}_{n-\ell}$.

Remark 5.8. The following holds.

- (1) For any $i \geq 1$, we have $\mathcal{Y}_{i,1} = \mathcal{M}_{i-1}$.
- (2) For any $2 \leq j \leq i$, we have $\mathcal{Y}_{i,j} = \Sigma\mathcal{Y}_{i-1,j-1}$.

In the rest of this article, we show that the pair $(\mathcal{M}_\ell, \mathcal{Y}_\ell)$ becomes a cotorsion pair for each $1 \leq \ell \leq n-1$. Since we have $\mathcal{B} = \mathcal{M}_n = \text{Cone}(\mathcal{M}_{n-1}, \mathcal{M})$, Lemma 5.6 and the dual of Proposition 4.1 shows that $(\mathcal{M}_1, \mathcal{Y}_1) = (\mathcal{M}, \mathcal{M}^{\perp_1})$ is a cotorsion pair.

First, let us show that $\mathcal{Y}_\ell \subseteq \mathcal{B}$ is closed under direct summands.

Lemma 5.9. *Let $\mathcal{X} \subseteq \mathcal{B}$ be a full additive subcategory closed by isomorphisms, containing \mathcal{I} . If $\mathcal{X} \subseteq \mathcal{B}$ is closed under direct summands and if $\mathbb{E}(\mathcal{I}, \mathcal{X}) = 0$ holds, then $\Sigma\mathcal{X} \subseteq \mathcal{B}$ is also closed under direct summands.*

Proof. Let $X \xrightarrow{x} I \xrightarrow{y} B_1 \oplus B_2 \xrightarrow{\delta} \rightarrow$ be any \mathbb{E} -triangle with $I \in \mathcal{I}$. It suffices to show that $X \in \mathcal{X}$ implies $B_1 \in \Sigma\mathcal{X}$. By $(\text{ET4})^{\text{op}}$, we have a commutative diagram made of \mathbb{E} -triangles

$$\begin{array}{ccccc} X & \xrightarrow{\exists g} & \exists X_1 & \longrightarrow & B_2 \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \delta} \rightarrow \\ \parallel & & \downarrow \exists m & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ X & \xrightarrow{x} & I & \xrightarrow{y} & B_1 \oplus B_2 \xrightarrow{\delta} \rightarrow \\ & & \downarrow z = \begin{pmatrix} 1 & 0 \end{pmatrix} y & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \\ & & B_1 & \xlongequal{\quad} & B_1 \\ & & \downarrow \exists \nu & & \downarrow 0 \\ & & \Psi & & \Psi \end{array} \quad (5.2)$$

satisfying $\begin{pmatrix} 1 & 0 \end{pmatrix}^* \nu = g_* \delta$. We will show $X_1 \in \mathcal{X}$. Realize $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \delta \in \mathbb{E}(B_1, X)$ as $X \xrightarrow{x'} Y \xrightarrow{y'} B_1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \delta} \rightarrow$. By Fact 1.19, we have the following commutative diagram made of \mathbb{E} -triangles

$$\begin{array}{ccccc} & & X_1 \xlongequal{\quad} X_1 & & \\ & & \downarrow \begin{pmatrix} \exists f \\ m \end{pmatrix} & & \downarrow m \\ X & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & X \oplus I & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & I \xrightarrow{0} \rightarrow \\ \parallel & & \downarrow \exists e & & \downarrow z \\ X & \xrightarrow{x'} & Y & \xrightarrow{y'} & B_1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \delta} \rightarrow \\ & & \downarrow y'^* \nu & & \downarrow \nu \\ & & \Psi & & \Psi \end{array}$$

satisfying

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_* \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \delta + \begin{pmatrix} f \\ m \end{pmatrix}_* \nu = 0, \quad (5.3)$$

in which, we may assume that the middle row is of the form $X \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \oplus I \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} I \xrightarrow{0} \rightarrow$ since we have $\mathbb{E}(I, X) = 0$ by assumption.

Since (5.3) implies $\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \delta = -f_* \nu$, we obtain a morphism of \mathbb{E} -triangles as follows.

$$\begin{array}{ccccccc} X_1 & \xrightarrow{m} & I & \xrightarrow{z} & B_1 & \xrightarrow{\nu} & \rightarrow \\ \downarrow -f & & \downarrow \exists_i & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \\ X & \xrightarrow{x} & I & \xrightarrow{y} & B_1 \oplus B_2 & \xrightarrow{\delta} & \rightarrow \end{array}$$

Composing this with a morphism of \mathbb{E} -triangles appearing in (5.2), we obtain the following.

$$\begin{array}{ccccccc} X_1 & \xrightarrow{m} & I & \xrightarrow{z} & B_1 & \xrightarrow{\nu} & \rightarrow \\ \downarrow -gf & & \downarrow i & & \parallel & & \\ X_1 & \xrightarrow{m} & I & \xrightarrow{z} & B_1 & \xrightarrow{\nu} & \rightarrow \end{array}$$

Then $(1 + gf)_* \nu = 0$ follows, and thus there is $j \in \mathcal{B}(I, X_1)$ which gives $1 + gf = jm$. This means that in the conflation

$$X_1 \xrightarrow{\begin{pmatrix} f \\ m \end{pmatrix}} X \oplus I \xrightarrow{e} Y, \quad (5.4)$$

the inflation $\begin{pmatrix} f \\ m \end{pmatrix}$ has a retraction $\begin{pmatrix} g & j \end{pmatrix}$, and thus (5.4) splits. In particular, we have $X_1 \in \mathcal{X}$. \square

Proposition 5.10. *For any $1 \leq j \leq i \leq n$, subcategory $\mathcal{Y}_{i,j} \subseteq \mathcal{B}$ is closed under direct summands. In particular, $\mathcal{Y}_\ell \subseteq \mathcal{B}$ is closed under direct summands, for any $1 \leq \ell \leq n$.*

Proof. We show by an induction on j . For $j = 1$, this follows from Remarks 5.6 and 5.8.

For $j > 1$, suppose that we have shown for $j - 1$. Let i be any integer satisfying $j \leq i \leq n$, and let us show that $\mathcal{Y}_{i,j} \subseteq \mathcal{B}$ is closed under direct summands. By the assumption of the induction, $\mathcal{Y}_{i-1,j-1} \subseteq \mathcal{B}$ is closed under direct summands. By definition, we have $\mathcal{I} \subseteq \mathcal{Y}_{i-1,j-1}$. Since $\mathbb{E}(\mathcal{M}, \mathcal{Y}_{i-1,j-1}) \cong \mathbb{E}^j(\mathcal{M}, \mathcal{M}_{i-j}) = 0$ by Lemma 5.6, we also have $\mathbb{E}(\mathcal{I}, \mathcal{Y}_{i-1,j-1}) = 0$. Thus Lemma 5.9 shows that $\mathcal{Y}_{i,j} = \Sigma \mathcal{Y}_{i-1,j-1} \subseteq \mathcal{B}$ is closed under direct summands. \square

Lemma 5.11. *For any $i, j \geq 0$, we have*

$$\mathbb{E}^k(\mathcal{M}_i, \mathcal{M}_j) = 0 \quad (i \leq k \leq n - j). \quad (5.5)$$

In particular, $\mathbb{E}(\mathcal{M}_\ell, \mathcal{Y}_\ell) = 0$ holds for any $1 \leq \ell \leq n$.

Proof. Let us show by an induction on i . For $i = 0$, this is trivial. For $i = 1$, this follows from Lemma 5.6.

For $i > 1$, suppose (5.5) holds for $i - 1$, for any $j \geq 0$. Let $X \in \mathcal{M}_i$ be any object, and take a conflation

$$M_{i-1} \rightarrowtail M \twoheadrightarrow X$$

with $M \in \mathcal{M}, M_{i-1} \in \mathcal{M}_{i-1}$. Then, since $\mathbb{E}^{k-1}(M_{i-1}, B) \rightarrow \mathbb{E}^k(X, B) \rightarrow \mathbb{E}^k(M, B)$ is exact for any $B \in \mathcal{B}$, the equality $\mathbb{E}^k(M, \mathcal{M}_j) = 0$ ($1 \leq k \leq n - j$) and the assumption of the induction

$$\mathbb{E}^{k-1}(M_{i-1}, \mathcal{M}_j) = 0 \quad (i - 1 \leq k - 1 \leq n - j)$$

shows $\mathbb{E}^k(X, \mathcal{M}_j) = 0$ for any $i \leq k \leq n - j$. \square

Lemma 5.12. *For any $1 \leq j \leq i$, we have $\mathcal{M}_i \subseteq \text{Cone}(\mathcal{Y}_{i,j}, \mathcal{M}_j)$. Especially for $i = n$,*

$$\mathcal{B} = \mathcal{M}_n = \text{Cone}(\mathcal{Y}_\ell, \mathcal{M}_\ell)$$

holds for any $1 \leq \ell \leq n - 1$.

Proof. We show by an induction on j . If $j = 1$, then for any $i \geq 1$, we have $\mathcal{M}_i = \text{Cone}(\mathcal{M}_{i-1}, \mathcal{M}) = \text{Cone}(\mathcal{Y}_{i,1}, \mathcal{M}_1)$ by definition.

For $j > 1$, suppose that we have shown for $j - 1$. Let i be any integer satisfying $j \leq i$, and let us show $\mathcal{M}_i \subseteq \text{Cone}(\mathcal{Y}_{i,j}, \mathcal{M}_j)$. Let $M_i \in \mathcal{M}_i$ be any object. By definition, there is a conflation

$$M_{i-1} \twoheadrightarrow M \twoheadrightarrow M_i \quad (M \in \mathcal{M}, M_{i-1} \in \mathcal{M}_{i-1}) \quad (5.6)$$

By the assumption of the induction, we have $\mathcal{M}_{i-1} \subseteq \text{Cone}(\mathcal{Y}_{i-1,j-1}, \mathcal{M}_{j-1})$. Thus there exists a conflation

$$Y \twoheadrightarrow M_{j-1} \twoheadrightarrow M_{i-1} \quad (Y \in \mathcal{Y}_{i-1,j-1}, M_{j-1} \in \mathcal{M}_{j-1}).$$

If we resolve Y by a conflation $Y \twoheadrightarrow I \twoheadrightarrow Y'$ with $I \in \mathcal{I}$, then we have $Y' \in \Sigma \mathcal{Y}_{i-1,j-1} = \mathcal{Y}_{i,j}$. By Fact 1.19, we have the following diagram made of conflations.

$$\begin{array}{ccccc} Y & \twoheadrightarrow & M_{j-1} & \twoheadrightarrow & M_{i-1} \\ \downarrow & & \downarrow & & \parallel \\ I & \twoheadrightarrow & \exists N & \twoheadrightarrow & M_{i-1} \\ \downarrow & & \downarrow & & \\ Y' & \xlongequal{\quad} & Y' & & \end{array}$$

Since the middle row splits, we may assume $N = I \oplus M_{i-1}$. From (5.6), we have a conflation

$$I \oplus M_{i-1} \twoheadrightarrow I \oplus M \twoheadrightarrow M_i.$$

Then by (ET4), we obtain the following commutative diagram made of conflations.

$$\begin{array}{ccccc} M_{j-1} & \twoheadrightarrow & I \oplus M_{i-1} & \twoheadrightarrow & Y' \\ \parallel & & \downarrow & & \downarrow \\ M_{j-1} & \twoheadrightarrow & I \oplus M & \twoheadrightarrow & \exists N' \\ & & \downarrow & & \downarrow \\ & & M_i & \xlongequal{\quad} & M_i \end{array}$$

Since $I \oplus M \in \mathcal{M}$, it follows $N' \in \text{Cone}(\mathcal{M}_{j-1}, \mathcal{M}) = \mathcal{M}_j$, and thus $M_i \in \text{Cone}(\mathcal{Y}_{i,j}, \mathcal{M}_j)$. \square

Lemma 5.13. $\mathcal{M}_\ell \cap \mathcal{Y}_\ell = \Sigma^{\ell-1} \mathcal{M}$ holds for any $1 \leq \ell \leq n-1$.

Proof. $\mathcal{M}_\ell \cap \mathcal{Y}_\ell \supseteq \Sigma^{\ell-1} \mathcal{M}$ follows immediately from the definition. Let us show $\mathcal{M}_\ell \cap \mathcal{Y}_\ell \subseteq \Sigma^{\ell-1} \mathcal{M}$ by an induction on ℓ . For $\ell = 1$, this follows from $\mathcal{M} \cap \mathcal{Y}_1 = \mathcal{M} \cap \mathcal{M}_{n-1} = \mathcal{M}$.

For $\ell > 1$, suppose that we have shown for $\ell - 1$. Let $N \in \mathcal{M}_\ell \cap \mathcal{Y}_\ell$ be any object. Since $N \in \mathcal{Y}_\ell = \Sigma^{\ell-1} \mathcal{M}_{n-\ell}$, there is a conflation

$$Z \twoheadrightarrow I \twoheadrightarrow N \quad (5.7)$$

with $Z \in \Sigma^{\ell-2} \mathcal{M}_{n-\ell} \subseteq \mathcal{Y}_{\ell-1}$ and $I \in \mathcal{I}$.

By Lemma 5.6, we have $\mathbb{E}(\mathcal{M}, Z) = 0$. Moreover, since (5.7) gives isomorphisms

$$\mathbb{E}^k(B, N) \cong \mathbb{E}^{k+1}(B, Z) \quad (\forall B \in \mathcal{B})$$

for any $k \geq 1$, we also obtain $\mathbb{E}^k(\mathcal{M}, Z) = 0$ for $2 \leq k \leq n - \ell + 1$ again by Lemma 5.6. Thus it follows $Z \in \mathcal{M}^{\perp_{n-\ell+1}} = \mathcal{M}_{\ell-1}$, and thus $Z \in \mathcal{M}_{\ell-1} \cap \mathcal{Y}_{\ell-1} = \Sigma^{\ell-2} \mathcal{M}$ by the assumption of the induction. Then (5.7) shows $N \in \Sigma^{\ell-1} \mathcal{M}$. \square

Theorem 5.14. Let $\mathcal{M} \subseteq \mathcal{B}$ be any n -cluster tilting subcategory. Then it induces a sequence of cotorsion pairs $(\mathcal{M}_1, \mathcal{Y}_1) \geq (\mathcal{M}_2, \mathcal{Y}_2) \geq \cdots \geq (\mathcal{M}_{n-1}, \mathcal{Y}_{n-1})$. Each cotorsion pair $(\mathcal{M}_\ell, \mathcal{Y}_\ell)$ has the core $\mathcal{W}_\ell = \Sigma^{\ell-1} \mathcal{M}$, the coheart $\mathcal{C}_\ell = \mathcal{M}$ and the kernel $\mathcal{K}_\ell = \mathcal{M}_{n-1}$.

Proof. By Lemma 5.6, Proposition 5.10, Lemmas 5.11, 5.12 and the dual of Proposition 4.1, the pair $(\mathcal{M}_\ell, \mathcal{Y}_\ell)$ becomes a cotorsion pair for any $1 \leq \ell \leq n-1$. Its core is given by Lemma 5.13. Since we already know ${}^{\perp 1}\mathcal{M}_{n-1} = \mathcal{M}$, it remains to show $\text{add}(\mathcal{M}_\ell * \mathcal{Y}_\ell) = \mathcal{M}_{n-1}$. For $\ell = 1$, this is obvious.

Let $2 \leq \ell \leq n-1$ be any integer. Then $\mathcal{M}_{n-1} \supseteq \text{add}(\mathcal{M}_\ell * \mathcal{Y}_\ell)$ follows from Lemma 5.6 and

$$\mathbb{E}(\mathcal{M}, \mathcal{M}_\ell) = 0, \quad \mathbb{E}(\mathcal{M}, \mathcal{Y}_\ell) \cong \mathbb{E}^\ell(\mathcal{M}, \mathcal{M}_{n-\ell}) = 0.$$

Conversely, let $M_{n-1} \in \mathcal{M}_{n-1}$ be any object. By Lemma 5.12, there is an \mathbb{E} -triangle

$$Y \rightarrow M_{\ell-1} \rightarrow M_{n-1} \dashrightarrow$$

with $Y \in \mathcal{Y}_{n-1, \ell-1}$, $M_{\ell-1} \in \mathcal{M}_{\ell-1}$. Resolve Y by an \mathbb{E} -triangle $Y \rightarrow I \rightarrow \Sigma Y \dashrightarrow$ with $I \in \mathcal{I}$. Then we obtain a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} Y & \longrightarrow & M_{\ell-1} & \longrightarrow & M_{n-1} \dashrightarrow \\ \parallel & & \downarrow & & \downarrow \\ Y & \longrightarrow & I & \longrightarrow & \Sigma Y \dashrightarrow \end{array}$$

since I is injective. By the dual of Proposition 1.20, we obtain a conflation $M_{\ell-1} \twoheadrightarrow M_{n-1} \oplus I \twoheadrightarrow \Sigma Y$, which means $M_{n-1} \in \text{add}(M_{\ell-1} * \Sigma \mathcal{Y}_{n-1, \ell-1}) \subseteq \text{add}(\mathcal{M}_\ell * \mathcal{Y}_\ell)$. \square

By Proposition 3.12 and Theorem 4.10, we get the following corollary.

Corollary 5.15. *By Proposition 3.12, all these $(\mathcal{M}_1, \mathcal{Y}_1), \dots, (\mathcal{M}_{n-1}, \mathcal{Y}_{n-1})$ are mutually heart-equivalent. Their hearts are equivalent to $\text{mod}(\mathcal{M}/\mathcal{P})$ by Theorem 4.10.*

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